APPLICATION OF SKEWNESS TO NON-LIFE RESERVING

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ABSTRACT

Since the 90's, there have been major achievements related to the quantification of the variance of the non-life reserves resulting from the application of different actuarial methods such as Chain-Ladder (Mack 1993a) and Bornhuetter-Ferguson (Mack 2008). These achievements have provided the basis for quantifying reserve risk in the different solvency frameworks currently in development (e.g. Solvency 2, Swiss Solvency Test). However, the choice of a risk distribution still remains to a large extent arbitrary as the knowledge of mean and variance allows the use of all available 2-parameter risk distributions (e.g. lognormal, Weibull, Pareto) without giving any preferences to any of them.

In order to select one parametric distribution, this paper proposes to estimate the third moment of the reserve risk distribution in a distribution-free environment. The quantification of higher moments and quantiles related to non-life reserving have already been treated using stochastic claim modeling. In particular, GLM models and bootstrap offer different possibilities of quantifying moments and quantiles (see Wüthrich-Merz (2008) and England and Verrall (2002)). However, these models require the specification of some explicit parametric distribution (e.g. for the residuals) in order to be applied. In contrast, this paper proposes a formulaic method to estimate the skewness of non-life reserves without requiring the specification of any particular distribution.

With the quantification of the skewness and knowing the mean and variance of the reserve risk distribution, this paper looks, on two examples, which of the lognormal distribution (which is the choice of the Quantitative Impact Study 5 in Solvency 2 for modeling reserve risk distribution) or of the gamma distribution fit best the calculated three moments. It concludes that, even though the gamma distribution fits better the three calculated moments, the lognormal distribution could still be used with the caveat that its skewness is higher and, as a consequence, the capital requirements in a Solvency 2 environment resulting from its use should be more conservative.

This paper is divided into the following sections:

- Section 1 provides basic definition of chain-ladder coefficients
- Section 2 looks into the definition of the standard error of reserve estimates
- Section 3 establishes the estimator of the skewness of the incurred claim amount for each development year of each accident year.
- Section 4 provides a description of the simulations done to estimate the skewness for non-life reserves for each accident year of a triangle.
- Section 5 provides numerical examples.

KEYWORDS

Skewness; Variance; Chain-Ladder; Reserve risk distribution; Correlation; Gaussian copula; Generalized Pareto Distribution; Generalized Extreme Value Distribution

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1. DEFINITIONS OF CHAIN-LADDER ELEMENTS

The following brief review of the Chain-Ladder method is based on Mack (1993a).

Let $C_{i,k}$ denote the cumulative incurred claims amount of accident year i after k years of development, $1 \le i, k \le I$, where I denotes the most recent accident year. Then $C_{I-i+1,i}$ denotes the currently known claims amount of accident year I-i+1. The basic chain-ladder assumption is that there exists development factors $f_1, \ldots, f_{I-1}$ such that:

$$E\left(C_{i,k+1} \mid C_{i,1},\ldots,C_{i,k}\right) = f_k C_{i,k}, 1 \le i \le I, 1 \le k \le I-1 \quad (1)$$

where the link ratios (age-to-age factors) can be estimated as follows:

$$\hat{f}_k = \frac{\sum_{j=1}^{I-k} C_{j,k+1}}{\sum_{j=1}^{I-k} C_{j,k}}, 1 \le k \le I\text{-}1 \quad (2)$$

under the assumption that $\{C_{i,1},\ldots,C_{i,I}\}, \{C_{j,1},\ldots,C_{j,I}\}, i \ne j$ are independent.

In all of this paper, $\hat{f}$ will denote the estimator of the random variable f .

Note: The $\hat{f}_k$ are unbiased and uncorrelated (see Mack (1993a)).

2. VARIANCE OF $C_{i,k}$

In the framework of distribution-free calculation of the standard error of the reserve estimates, several variance models exist. For the purpose of this article, we will focus on the Mack standard error.

As for the variance of $C_{i,k+1}$, T. Mack (1993a) induced that $Var\left(C_{i,k+1} \mid C_{i,1},\ldots,C_{i,k}\right)$ (where $Var(A \mid B)$ denotes the conditional variance of A knowing B) should be proportional to $C_{i,k}$,

i.e.: $Var\left(C_{i,k+1} \mid C_{i,1},\ldots,C_{i,k}\right) = C_{i,k}\sigma_k^2 , 1 \le i \le I, 1 \le k \le I\text{-}1$ (3)

where $\hat{\sigma}_k^2 = \frac{1}{I-k-1}\sum_{i=1}^{I-k} C_{i,k}\left(\frac{C_{i,k+1}}{C_{i,k}} - \hat{f}_k\right)^2 \quad for\ 1 \le k \le I\text{-}2$ (4)

It can be shown that the estimator $\hat{\sigma}_k^2$ is unbiased (see Mack (1993a)).

Using the Mack variance model, the next section will focus on finding an expression of the skewness of $C_{i,k}$.

3. SKEWNESS OF $C_{i,k}$

The aim of this section is to find an expression for the skewness of $C_{i,k}$ within the framework of the Mack variance model.

Let's denote $SK(A\,|\,B)$, the conditional skewness of A knowing B:

$$SK(A\,|\,B) = E\left[(A - E(A\,|\,B))^3\,|\,B\right]$$

This expression is different from the usual definition of the skewness (denoted hereinafter by the abbreviation "Skew"):

$$Skew(A\,|\,B) = \frac{E\left[(A - E(A\,|\,B))^3\,|\,B\right]}{Var(A\,|\,B)^{3/2}}$$

but it will prove to be useful.

For the purpose of this article, it will be assumed that:

$$Skew(C_{i,k+1}\,|\,C_{i,1},\dots,C_{i,k}) = \frac{SK(C_{i,k+1}\,|\,C_{i,1},\dots,C_{i,k})}{\left[Var(C_{i,k+1}\,|\,C_{i,1},\dots,C_{i,k})\right]^{3/2}}$$ depends on k but does not depend on i.

This assumption means that, for one development year within a triangle, the asymmetry (denoted "Skew") is the same for any accident year. This assumption can be applied in the context of a reserving portfolio which risks would have the same characteristics for every accident year. Before applying the results of this article, this assumption should therefore be checked. This check can be done on plotting $\frac{\left(C_{i,k+1} - \hat{f}_k C_{i,k}\right)^3}{C_{i,k}^{3/2}}$ against $C_{i,k}^{3/2}$ (as $\left[Var(C_{i,k+1}\,|\,C_{i,1},\dots,C_{i,k})\right]^{3/2} = \left(\hat{\sigma}_k^2 C_{ik}\right)^{3/2}$) for each i within one development year k, and on verifying that the resulting plot does not show any trend depending on i. Such graphs are shown in appendix 3 for the chosen numerical examples.

With such an assumption, we have:

$$\exists \gamma_k \text{ such that } \gamma_k = \frac{SK(C_{i,k+1}\,|\,C_{i,1},\dots,C_{i,k})}{\left[Var(C_{i,k+1}\,|\,C_{i,1},\dots,C_{i,k})\right]^{3/2}} = \frac{SK(C_{i,k+1}\,|\,C_{i,1},\dots,C_{i,k})}{\left[\sigma_k^2 C_{i,k}\right]^{3/2}}$$

$$\Rightarrow SK(C_{i,k+1}\,|\,C_{i,1},\dots,C_{i,k}) = \gamma_k \left[\sigma_k^2 C_{i,k}\right]^{3/2}$$

Hence, we have the result that $SK(C_{i,k+1}\,|\,C_{i,1},\dots,C_{i,k})$ is proportional to $C_{i,k}^{3/2}$, i.e. :

$$SK(C_{i,k+1}\,|\,C_{i,1},\dots,C_{i,k}) = C_{i,k}^{3/2} Sk_k^3 \,,\, 1 \le i \le I,\, 1 \le k \le I\text{-}2 \tag{5}$$

Based on the definition above, the Theorem 1 below will give an unbiased estimator for Sk_k^3 .

Theorem 1:
Similar to the variance expression in equation (4), we will show that the estimator below is unbiased.

$$\hat{S}k_k^3 = \frac{1}{\left(I - k - \frac{\left(\sum_{i=1}^{I-k} C_{i,k}^{3/2}\right)^2}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^3} \right)} \sum_{i=1}^{I-k} C_{i,k}^{3/2} \left(\frac{C_{i,k+1}}{C_{i,k}} - \hat{f}_k \right)^3 \quad \text{for } 1 \leq k \leq I\text{-}2 \tag{6}$$

For $k = I\text{-}1$, as data is not available to estimate $\hat{S}k_k^3$, it will be assumed that $\hat{S}k_k^3 = 0$ for this article.

The proof of this theorem is detailed in Appendix 1.

Note on the estimator $\hat{S}k_k^3$:

This estimator will have a high value if, for some i, $\frac{\left(C_{i,k+1} - \hat{f}_k C_{i,k}\right)^3}{C_{i,k}^{3/2}}$ has a high value.

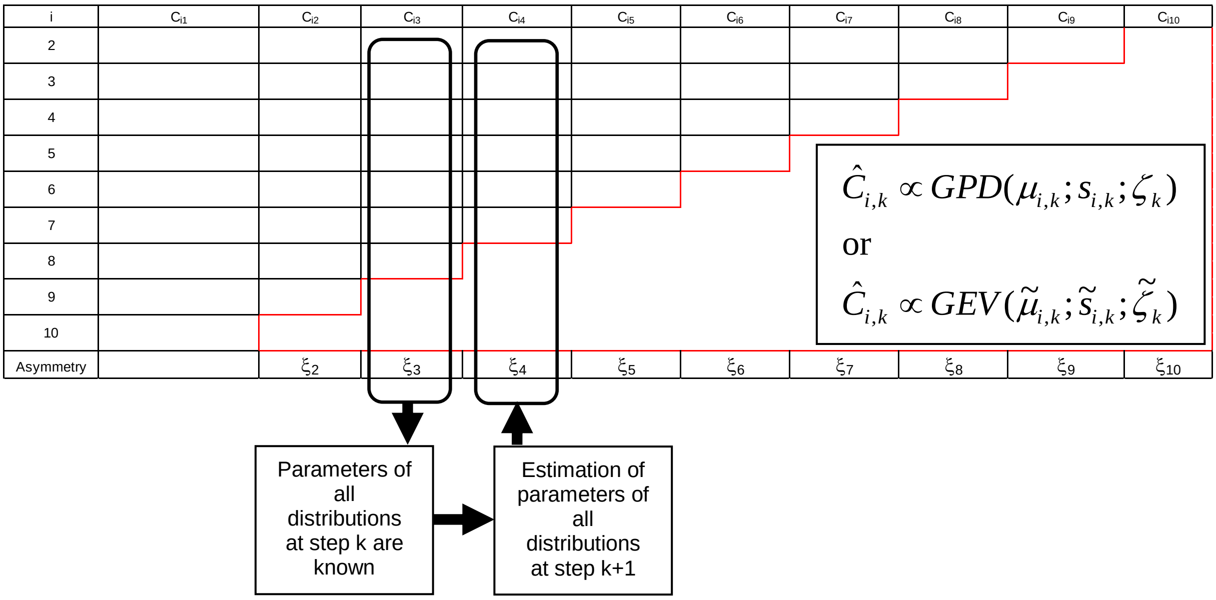
This would happen in the case of an "outlier" in the triangle data. In this case, an "outlier" is defined as a jump in one accident year from $C_{i,k+1}$ in comparison to the expected amount $\hat{f}_k C_{i,k}$. In addition, in order to have a high skewness, it is necessary to have jumps of the same sign and not compensating each other. Having recognized such mechanism, the actuary needs to use his judgment to establish how reliable the "outliers" in a triangle are. In general, this evaluation is important for estimating the standard error of the loss reserves but it is even more important to estimate the skewness within the proposed framework.

4. SKEWNESS OF NON-LIFE RESERVES BY ACCIDENT YEAR

In order to estimate the skewness of non-life reserves by accident year, it is necessary to create a stochastic model. This model will have the characteristics described in this paragraph so that the best estimate and standard deviations match the moments coming from the Mack model (1993a). For the triangle below, we will fit the $\hat{C}_{i,k}$ in the lower part successively with two types of distribution:

- a Generalized Pareto Distribution (hereinafter "GPD") with parameters $(\mu_{i,k};s_{i,k};\zeta_k)$ with $\mu_{i,k} \in (-\infty;\infty)$, the location parameter, $s_{i,k} \in (0;\infty)$, the scale parameter and $\zeta_k \in (-\infty;\infty)$, the shape parameter (we will see below that this shape parameter is related to the asymmetry of this distribution, which according to paragraph 3 depends only on k).
- a Generalized Extreme Value distribution (hereinafter "GEV") with parameters $(\tilde{\mu}_{i,k};\tilde{s}_{i,k};\tilde{\zeta}_k)$ with $\tilde{\mu}_{i,k} \in (-\infty;\infty)$, the location parameter, $\tilde{s}_{i,k} \in (0;\infty)$, the scale parameter and $\tilde{\zeta}_k \in (-\infty;\infty)$, the shape parameter (like for the GPD distribution, $\tilde{\zeta}_k$ depends only on k).

The stochastic model is iterative: At step k (corresponding to the column k of the below triangle), all the parameters are known. On the basis of these parameters known at step k, the parameters of step k+1 are estimated using the formulas shown in the following sections.



Note: The chosen distribution (GEV or GPD) is merely a medium to use the known skewness $\hat{Sk}_k^3$ of a development year so as to estimate an accident year skewness. In principle, the choice of the distribution should not have an impact on the total skewness.

4.1 Model GPD

For the GPD model, with the above parameters, we have the following results:

- $\hat{M}ean_{i,k} = \mu_{i,k} + \frac{s_{i,k}}{1-\zeta_k} = E\left(\hat{C}_{i,k}\right)$
- $\hat{V}ariance_{i,k} = \frac{s_{i,k}^2}{(1-\zeta_k)^2(1-2\zeta_k)} = Var\left(\hat{C}_{i,k}\right)$ (with $\zeta_k < 1/2$)
- $Skew\left(\hat{C}_{i,k}\right) = \xi_k = \frac{SK\left(\hat{C}_{i,k}\right)}{Var\left(\hat{C}_{i,k}\right)^{3/2}} = \frac{2(1+\zeta_k)\sqrt{1-2\zeta_k}}{1-3\zeta_k}$

At step k of the stochastic run (which corresponds to a development year in the triangle above), assuming that we know the 3 parameters $(\mu_{i,k}; s_{i,k}; \zeta_k)$, we can generate the random variable $C_{i,k}$ with the following method:

- Draw a random variable $U_{i,k}$ uniformly distributed on $(0,1]$. We will see later in this section that there is a dependency structure between the $U_{i,k}$ which follows a Gaussian copula that needs to be specified to match the assumptions of Mack (1993a).
- Then: $\hat{C}_{i,k} = \mu_{i,k} + \frac{s_{i,k}\left(U_{i,k}^{-\zeta_k} - 1\right)}{\zeta_k} \propto GPD(\mu_{i,k}; s_{i,k}; \zeta_k)$

At step k+1, we have the following formulas:

- $\xi_{k+1} = \frac{\hat{S}k_{k+1}^3}{\hat{\sigma}_{k+1}^3} = \frac{2(1+\zeta_{k+1})\sqrt{1-2\zeta_{k+1}}}{1-3\zeta_{k+1}}$ which allows to estimate the parameter ζ_{k+1}.
- $\hat{V}ariance_{i,k+1} = \hat{C}_{i,k}\hat{\sigma}_k^2\left(1 + \frac{\hat{C}_{i,k}}{\sum_{j=1}^{I-k} C_{j,k}}\right)$. This definition indicated in Mack (1999) allows for a programmable recursive way of calculating the standard error of the chain-ladder reserve estimates shown in Mack (1993a).

  The knowledge of $\hat{V}ariance_{i,k+1}$ and of the parameter ζ_{k+1} allows the estimation of the parameter $s_{i,k+1}$.
- Finally, we have: $\hat{C}_{i,k+1} = \hat{f}_k\hat{C}_{i,k} = \mu_{i,k+1} + \frac{s_{i,k+1}}{1-\zeta_{k+1}}$ which allows the estimation of the parameter $\mu_{i,k+1}$.

4.2 Model GEV

For the GEV model, with the above parameters, we have the following results:

- $\hat{Mean}_{i,k} = \tilde{\mu}_{i,k} + \tilde{s}_{i,k} \dfrac{\Gamma\left(1+\tilde{\zeta}_k\right)-1}{\tilde{\zeta}_k}$
- $\hat{Variance}_{i,k} = \tilde{s}_{i,k}^2 \dfrac{\Gamma\left(1-2\tilde{\zeta}_k\right)-\Gamma\left(1-\tilde{\zeta}_k\right)^2}{\tilde{\zeta}_k^{\,2}}$
- $\xi_k = \dfrac{SK\left(\hat{C}_{i,k}\right)}{Var\left(\hat{C}_{i,k}\right)^{3/2}} = \dfrac{\Gamma\left(1-3\tilde{\zeta}_k\right)-3\Gamma\left(1-\tilde{\zeta}_k\right)\Gamma\left(1-2\tilde{\zeta}_k\right)+2\Gamma\left(1-\tilde{\zeta}_k\right)^3}{\left(\Gamma\left(1-2\tilde{\zeta}_k\right)-\Gamma\left(1-\tilde{\zeta}_k\right)^2\right)^{3/2}}$

At step k of the stochastic run, assuming that we know the 3 parameters $(\tilde{\mu}_{i,k};\tilde{s}_{i,k};\tilde{\zeta}_k)$, we can generate the random variable $C_{i,k}$ with the following method:

- Draw a random variable $U_{i,k}$ uniformly distributed on $\left(0,1\right]$. The same dependency structure between the $U_{i,k}$ applied in the GPD model is used for the GEV model.
- Then: $\hat{C}_{i,k} = \tilde{\mu}_{i,k} + \dfrac{\tilde{s}_{i,k}\left(\left(-\ln\left(U_{i,k}\right)\right)^{-\tilde{\zeta}_k}-1\right)}{\tilde{\zeta}_k} \propto GEV(\tilde{\mu}_{i,k};\tilde{s}_{i,k};\tilde{\zeta}_k)$

At step k+1, we have the following formulas:

- $\xi_{k+1} = \dfrac{\hat{S}k_{k+1}^3}{\hat{\sigma}_{k+1}^3} = \dfrac{\Gamma\left(1-3\tilde{\zeta}_{k+1}\right)-3\Gamma\left(1-\tilde{\zeta}_{k+1}\right)\Gamma\left(1-2\tilde{\zeta}_{k+1}\right)+2\Gamma\left(1-\tilde{\zeta}_{k+1}\right)^3}{\left(\Gamma\left(1-2\tilde{\zeta}_{k+1}\right)-\Gamma\left(1-\tilde{\zeta}_{k+1}\right)^2\right)^{3/2}}$ which allows to estimate the parameter $\tilde{\zeta}_{k+1}$.
- $\hat{Variance}_{i,k+1} = \hat{C}_{i,k}\hat{\sigma}_k^2\left(1+\dfrac{\hat{C}_{i,k}}{\sum_{j=1}^{I-k} C_{j,k}}\right)$. The knowledge of $\hat{Variance}_{i,k+1}$ and of the parameter $\tilde{\zeta}_{k+1}$ allows the estimation of the parameter $\tilde{s}_{i,k+1}$.
- Finally, we have: $\hat{C}_{i,k+1} = \hat{f}_k\hat{C}_{i,k} = \tilde{\mu}_{i,k+1} + \tilde{s}_{i,k+1}\dfrac{\Gamma\left(1+\tilde{\zeta}_{k+1}\right)-1}{\tilde{\zeta}_{k+1}}$. This allows the estimation of the parameter $\tilde{\mu}_{i,k+1}$.

With a stepwise simulation, it is then possible to have the overall reserve risk distribution. However, this simulation requires the definition of the dependency structure between the $U_{i,k}$ at each step of the simulation. This is the object of the section below.

4.3 Gaussian copula $U_{i,k}$:

According to Mack (1993a), the mse of the overall reserve estimate $\hat{R} = \hat{R}_2 + ... + \hat{R}_I$ is equal to:

$$\hat{m}se(\hat{R}) = \sum_{i=2}^{I} \left\{ \hat{m}se\left(\hat{R}_i\right) + \hat{C}_{i,I} \left(\sum_{j=i+1}^{I} \hat{C}_{j,I} \right) \sum_{k=I+1-i}^{I-1} \frac{2\hat{\sigma}_k^2 \big/ \hat{f}_k^2}{\sum_{n=1}^{I-k} C_{n,k}} \right\}$$

This equation includes an embedded assumptions of correlation between the mse of the reserve estimates $\hat{R}_i$. A natural way to simulate such correlations is to use a Gaussian copula. A definition and some properties of the Gaussian copula are given in Appendix 2.

Re-writing the above equation in the usual way, we have:

$$\hat{m}se(\hat{R}) = \sum_{i=2}^{I} \hat{m}se\left(\hat{R}_i\right) + 2 \sum\sum_{i<j} \rho_{i,j} \sqrt{\hat{m}se\left(\hat{R}_i\right) \hat{m}se\left(\hat{R}_j\right)}$$

And we can see that the correlations between mse are:

$$\rho_{i,j} = \frac{\hat{C}_{i,I} \hat{C}_{j,I} \sum_{k=I+1-i}^{I-1} \frac{\hat{\sigma}_k^2 \big/ \hat{f}_k^2}{\sum_{n=1}^{I-k} C_{n,k}}}{\sqrt{\hat{m}se\left(\hat{R}_i\right) \hat{m}se\left(\hat{R}_j\right)}}$$

These correlations are valid at the ultimate. In the case of our stepwise simulation, it is necessary to have a correction on these correlations so as to reflect the fact that, at step k, the view is not yet on the ultimate. This correction is simply done as follows:

$$\tilde{\rho}_{i,j} = \rho_{i,j} \prod_{n=I-i+1}^{I-1} \hat{f}_n^2 \text{ where } i < j .$$

These correlations are the characteristics of the Gaussian copula that is used in our stepwise simulation.

5. NUMERICAL EXAMPLE

5.1 Example 1

To demonstrate the above method by means of a simple example, the methodology is applied to the first triangle used in Mack (1993a):

i	C_{i1}	C_{i2}	C_{i3}	C_{i4}	C_{i5}	C_{i6}	C_{i7}	C_{i8}	C_{i9}	C_{i10}
1	357'848	1'124'788	1'735'330	2'218'270	2'745'596	3'319'994	3'466'336	3'606'286	3'833'515	3'901'463
2	352'118	1'236'139	2'170'033	3'353'322	3'799'067	4'120'063	4'647'867	4'914'039	5'339'085	
3	290'507	1'292'306	2'218'525	3'235'179	3'985'995	4'132'918	4'628'910	4'909'315		
4	310'608	1'418'858	2'195'047	3'757'447	4'029'929	4'381'982	4'588'268			
5	443'160	1'136'350	2'128'333	2'897'821	3'402'672	3'873'311				
6	396'132	1'333'217	2'180'715	2'985'752	3'691'712					
7	440'832	1'288'463	2'419'861	3'483'130						
8	359'480	1'421'128	2'864'498							
9	376'686	1'363'294								
10	344'014									

Table 1: Triangle of cumulative incurred claims

For this triangle, I = 10. It is possible to calculate the statistics $\hat{f}_k$, $\hat{\sigma}_k^2$, $\hat{Sk}_k^3$, ζ_k and ξ_k:

k	1	2	3	4	5	6	7	8	9
$\hat{f}_k$	3.491	1.747	1.457	1.174	1.104	1.086	1.054	1.077	1.018
$\hat{\sigma}_k^2$	160'280	37'737	41'965	15'183	13'731	8'186	447	1'147	447
$\hat{Sk}_k^3$	8'813'962	1'578'617	5'482'361	-810'454	646'781	-19'063	-4'691	-4'239	NA
ξ_κ	0.137	0.215	0.638	-0.433	0.402	-0.026	-0.497	-0.109	NA
GPD ζ_κ	-0.851	-0.776	-0.455	-1.616	-0.617	-1.030	-1.728	-1.133	-1.000
GEV ζ_κ	-0.235	-0.211	-0.100	-0.427	-0.159	-0.286	-0.450	-0.313	-0.278

Table 2: Chain-ladder coefficient, mack variance, skewness of each development year

It is interesting to note that $\hat{Sk}_4^3$ is negative. Looking at the triangle, we can see that this negative value is mainly driven by the accident year 4 where there seems to be an unusual development in the incurred claims. As a consequence, the chain-ladder coefficients for this accident year compared to the chain-ladder coefficients of the other accident years seem to show a different pattern:

i/k	1	2	3	4	5	6	7	8	9
1	3.143	1.543	1.278	1.238	1.209	1.044	1.040	1.063	1.018
2	3.511	1.755	1.545	1.133	1.084	1.128	1.057	1.086	
3	4.448	1.717	1.458	1.232	1.037	1.120	1.061		
4	4.568	**1.547**	**1.712**	**1.073**	1.087	1.047			
5	2.564	1.873	1.362	1.174	1.138				
6	3.366	1.636	1.369	1.236					
7	2.923	1.878	1.439						
8	3.953	2.016							
9	3.619								

Table 3: Triangle of chain-ladder coefficients by accident year and development year

For the accident year 4, the unusual development seems to have started in the development year 2 and to have had repercussions until development year 4. This could have come from a big claim which would have been integrated in this triangle and which has modified the usual chain-ladder development patterns. For the purpose of this article, we will use the figures that result from the application of formula (6) bearing in mind however that such figures should have been corrected with the knowledge of the underlying claims data.

As a final data input, the correlation matrix of the $\tilde{\rho}_{i,j}$ used for the Gaussian copula is given below:

i / j	2	3	4	5	6	7	8	9	10
2	100.0%	35.8%	32.1%	15.0%	10.1%	8.2%	6.3%	4.7%	2.9%
3	35.8%	100.0%	40.1%	18.8%	12.6%	10.2%	7.8%	5.9%	3.7%
4	32.1%	40.1%	100.0%	19.2%	12.8%	10.5%	8.0%	6.0%	3.8%
5	15.0%	18.8%	19.2%	100.0%	16.1%	13.1%	10.0%	7.5%	4.7%
6	10.1%	12.6%	12.8%	16.1%	100.0%	16.7%	12.7%	9.6%	6.0%
7	8.2%	10.2%	10.5%	13.1%	16.7%	100.0%	15.1%	11.3%	7.1%
8	6.3%	7.8%	8.0%	10.0%	12.7%	15.1%	100.0%	14.6%	9.1%
9	4.7%	5.9%	6.0%	7.5%	9.6%	11.3%	14.6%	100.0%	9.4%
10	2.9%	3.7%	3.8%	4.7%	6.0%	7.1%	9.1%	9.4%	100.0%

Table 4: Correlation matrix $\rho_{i,j}$

The results of 10 000 simulations following the process described in section 4 are shown below:

	Actual data		GPD model					GEV model				
i	Mean	Mack mse	Mean	Difference to actual	Mse	Difference to actual	Skewness	Mean	Difference to actual	Mse	Difference to actual	Skewness
2	94'634	75'535	95'165	-0.6%	75'470	0.1%	-0.0116	95'386	-0.8%	75'592	-0.1%	-0.0308
3	469'511	121'699	471'151	-0.3%	119'704	1.7%	0.0481	471'525	-0.4%	122'739	-0.8%	0.0009
4	709'638	133'549	710'157	-0.1%	132'391	0.9%	0.0293	708'230	0.2%	133'866	-0.2%	0.0082
5	984'889	261'406	981'736	0.3%	262'649	-0.5%	-0.2775	982'886	0.2%	263'559	-0.8%	-0.2852
6	1'419'459	411'010	1'422'432	-0.2%	414'718	-0.9%	-0.0447	1'414'641	0.3%	412'495	-0.4%	-0.0557
7	2'177'641	558'317	2'175'447	0.1%	560'128	-0.3%	0.1760	2'176'702	0.0%	556'291	0.4%	0.1734
8	3'920'301	875'328	3'922'171	0.0%	877'225	-0.2%	0.0372	3'926'519	-0.2%	869'263	0.7%	0.0605
9	4'278'972	971'258	4'291'596	-0.3%	970'565	0.1%	0.2796	4'292'016	-0.3%	981'608	-1.1%	0.2742
10	4'625'811	1'363'155	4'653'011	-0.6%	1'360'071	0.2%	0.3546	4'627'128	0.0%	1'371'339	-0.6%	0.4359
Total	18'680'856	2'447'095	18'722'867	-0.2%	2'356'529	3.8%	0.1385	18'695'034	-0.1%	2'358'315	3.8%	0.1475

Table 5: Simulation results

The above results show that the GPD model and the GEV model are in line with the Mack model in terms of mean squared error ("mse") and in terms of mean. In addition, it is interesting to note that the skewness per accident year are also consistent between both models. Finally, we can compare the total skewness resulting from either models with the skewness which would result from the fitting of the total mean and mse to a gamma distribution and to a lognormal distribution:

Lognormal		Gamma	
μ	16.735	k	58.276
σ	0.130	θ	320'556.691
Skewness	0.395	0.262	

Table 6: Parameters of Lognormal and Gamma distributions

This comparison shows that the simulated total skewness (0.1385 in the GPD model vs 0.1475 in the GEV model) is lower than the skewness which would stem from either a gamma model (0.262) or a lognormal model (0.395).

As a final result, we compare the value at risk at 99% coming from the simulations with the value at risk at 99% which would result from the fitting of the total mean and mse to a gamma distribution and to a lognormal distribution:

Lognormal		Gamma		Simulation GPD	Simulation GEV
μ	16.735	k	58.276		
σ	0.130	θ	320'556.691		
VaR 99%	25'089'172		24'840'064	24'481'480	24'277'909

Table 7: Comparison of Value at Risk 99%

As anticipated due to the lower skewness of the simulated distributions, the simulated VaR 99% are lower than those coming from the applications of a lognormal or of a gamma distribution.

5.2 Example 2

We will also use this second triangle to illustrate the methodology presented in this article.

i	C_{i1}	C_{i2}	C_{i3}	C_{i4}	C_{i5}	C_{i6}	C_{i7}	C_{i8}	C_{i9}	C_{i10}
1	397	793	868	889	964	964	964	964	964	964
2	523	1'098	1'475	1'489	1'489	1'489	1'489	1'489	1'489	
3	1'786	2'951	3'370	3'029	3'211	3'289	3'325	3'325		
4	241	465	536	596	652	652	652			
5	327	622	577	583	583	583				
6	275	520	529	529	541					
7	89	327	378	382						
8	295	301	396							
9	151	406								
10	315									

Table 8: Triangle of cumulative incurred claims

For this triangle, I = 10. The statistics $\hat{f}_k$, $\hat{\sigma}_k^2$, $\hat{S}k_k^3$, ζ_k and ξ_k are given below:

k	1	2	3	4	5	6	7	8	9
$\hat{f}_k$	1.832	1.149	0.969	1.046	1.011	1.006	1.000	1.000	1.000
$\hat{\sigma}_k^2$	90	13	6	2	0	0	0	0	0
$\hat{S}k_k^3$	401	12	-3	-1	0	0	0	0	NA
ξ_κ	0.472	0.256	-0.249	-0.285	0.378	0.165	0.773	0.153	NA
GPD ζ_κ	-0.565	-0.738	-1.325	-1.377	-0.635	-0.824	-0.378	-0.836	-1.000
GEV ζ_κ	-0.141	-0.199	-0.361	-0.374	-0.165	-0.226	-0.070	-0.230	-0.278

Table 9: Chain-ladder coefficient, mack variance, skewness of each development year

The correlation matrix of the $\rho_{i,j}$ used for the Gaussian copula is given below:

i / j	2	3	4	5	6	7	8	9	10
2	100.0%	51.5%	29.8%	0.5%	0.2%	0.1%	0.0%	0.0%	0.0%
3	51.5%	100.0%	39.7%	0.6%	0.3%	0.1%	0.0%	0.0%	0.0%
4	29.8%	39.7%	100.0%	0.4%	0.2%	0.1%	0.0%	0.0%	0.0%
5	0.5%	0.6%	0.4%	100.0%	3.7%	1.3%	0.6%	0.4%	0.3%
6	0.2%	0.3%	0.2%	3.7%	100.0%	2.6%	1.3%	0.9%	0.5%
7	0.1%	0.1%	0.1%	1.3%	2.6%	100.0%	2.5%	1.7%	1.0%
8	0.0%	0.0%	0.0%	0.6%	1.3%	2.5%	100.0%	3.4%	2.1%
9	0.0%	0.0%	0.0%	0.4%	0.9%	1.7%	3.4%	100.0%	3.3%
10	0.0%	0.0%	0.0%	0.3%	0.5%	1.0%	2.1%	3.3%	100.0%

Table 10: Correlation matrix $\rho_{i,j}$

The results of 10 000 simulations following the process described in section 4 are shown below:

	Actual data		GPD model					GEV model				
i	Mean	Mack mse	Mean	Difference to actual	Mse	Difference to actual	Skewness	Mean	Difference to actual	Mse	Difference to actual	Skewness
2	0	0	0	0.1%	0	0.0%	0.0056	0	-0.1%	0	-0.2%	-0.0115
3	0	0	0	0.1%	0	1.5%	0.0154	0	-1.0%	0	-0.3%	0.0466
4	0	0	0	0.1%	0	-1.0%	0.0200	0	0.5%	0	-0.4%	0.0168
5	3	6	3	-1.4%	6	-0.6%	0.7871	3	0.0%	6	-0.9%	0.7916
6	9	14	9	-1.1%	14	0.4%	0.2016	9	-0.8%	14	-0.2%	0.2102
7	24	28	24	2.9%	28	0.8%	0.3442	25	-1.5%	28	0.3%	0.2953
8	12	59	12	0.6%	59	0.4%	-0.0768	12	0.5%	60	-0.2%	-0.0383
9	75	101	76	-1.1%	101	0.2%	0.0133	75	-0.3%	103	-2.3%	0.0246
10	369	240	370	-0.3%	238	0.8%	0.3774	369	-0.1%	239	0.5%	0.3694
Total	494	274	495	-0.2%	270	1.8%	0.2702	494	-0.2%	271	1.2%	0.2314

Table 11: Simulation results

As for example 1, we can compare the total skewness resulting from either models with the skewness which would result from the fitting of the total mean and mse to a gamma distribution and to a lognormal distribution:

Lognormal		Gamma	
μ	6.067	k	3.234
σ	0.519	θ	152.638
Skewness	1.840	1.112	

As for example 1, this comparison shows that the simulated total skewness for both GPD and GEV models is lower than the skewness which would stem from either a gamma model or a lognormal model. As a final result, we compare the value at risk at 99% coming from the simulations with the value at risk at 99% which would result from the fitting of the total mean and mse to a gamma distribution and to a lognormal distribution:

Lognormal		Gamma		Simulation GPD	Simulation GEV
μ	6.067	k	3.234		
σ	0.519	θ	152.638		
VaR 99%	1'443	1'343		1'152	1'182

As anticipated due to the lower skewness of the simulated distributions, the simulated VaR 99% are lower than those coming from the applications of a lognormal or of a gamma distribution.

6. CONCLUSION

In this paper, the skewness information is used for the purpose of determination of non-life reserve risk distribution. This paper shows that the knowledge of the skewness provides valuable insights into the non-life ultimate reserves of an insurance or a reinsurance company. In particular, on the basis of 2 examples, it is clear that the lognormal distribution used for the solvency 2 standard model is a conservative choice; the gamma distribution would seem to fit better, even though it would remain a conservative choice. In order to further enhance the knowledge of the reserve risk distribution, a study of higher moments, in particular, the kurtosis operator should be done.

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APPENDIX 1 – Proof of theorem 1

We want to show that $E\left(\hat{S}k_k^3\right) = Sk_k^3$. Therefore, we will, in a first step, concentrate on $E\left(\hat{S}k_k^3 \mid B_k\right)$.

We have:

$$(Eq) = E\left[\left(I-k-\frac{\left(\sum_{i=1}^{I-k} C_{i,k}^{3/2}\right)^2}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^3}\right)\hat{S}k_k^3 \mid B_k\right] = E\left[\sum_{i=1}^{I-k} C_{i,k}^{3/2}\left(\frac{C_{i,k+1}}{C_{i,k}} - \hat{f}_k\right)^3 \mid B_k\right] \text{ for } 1 \le k \le I\text{-}3$$

where $B_k = \left\{C_{i,j} \mid i+j \le I+1, j \le k\right\}, 1 \le k \le I\text{-}1$

$$(Eq) = \sum_{i=1}^{I-k}\left[\frac{1}{C_{i,k}^{3/2}} E\left(C_{i,k+1}^3 \mid B_k\right)\right] - 3\sum_{i=1}^{I-k}\left[\frac{1}{\sqrt{C_{i,k}}} E\left(\hat{f}_k C_{i,k+1}^2 \mid B_k\right)\right] + 3\sum_{i=1}^{I-k}\left[\sqrt{C_{i,k}}\, E\left(\hat{f}_k^2 C_{i,k+1} \mid B_k\right)\right] - \sum_{i=1}^{I-k} C_{i,k}^{3/2} E\left(\hat{f}_k^3 \mid B_k\right) \tag{7}$$

For each of the 4 elements of the above equation, a formula is going to be established. For the first element of equation (7), we have:

$$SK\left(C_{i,k+1} \mid B_k\right) = Sk_k^3 C_{i,k}^{3/2} = E\left(C_{i,k+1}^3 \mid B_k\right) - 3E\left(C_{i,k+1} \mid B_k\right)E\left(C_{i,k+1}^2 \mid B_k\right) + 2E\left(C_{i,k+1} \mid B_k\right)^3$$

From Mack (1993b), we have:

$E\left(C_{i,k+1}^2 \mid B_k\right) = C_{i,k}\sigma_k^2 + C_{i,k}^2 f_k^2$ and

$E\left(C_{i,k+1} \mid B_k\right) = C_{i,k} f_k$

Therefore:

$$E\left(C_{i,k+1}^3 \mid B_k\right) = Sk_k^3 C_{i,k}^{3/2} + 3 f_k C_{i,k}^2 \sigma_k^2 + C_{i,k}^3 f_k^3 \tag{8}$$

For the fourth element of equation (7), we have:

$$SK\left(\hat{f}_k \mid B_k\right) = \frac{1}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^3} \sum_{i=1}^{I-k} SK\left(C_{i,k+1} \mid B_k\right) = \frac{Sk_k^3}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^3} \sum_{i=1}^{I-k} C_{i,k}^{3/2}$$

$$= E\left(\left[\hat{f}_k - E\left(\hat{f}_k \mid B_k\right)\right]^3 \mid B_k\right) = E\left(\hat{f}_k^3 \mid B_k\right) - 3E\left(\hat{f}_k \mid B_k\right)E\left(\hat{f}_k^2 \mid B_k\right) + 2E\left(\hat{f}_k \mid B_k\right)^3$$

From Mack (1993b), we have:

$$E\left(\hat{f}_k^2 \mid B_k\right) = \frac{\sigma_k^2}{\sum_{i=1}^{I-k} C_{i,k}} + f_k^2$$

Therefore:

$$E\left(\hat{f}_k^3 \mid B_k\right) = \frac{Sk_k^3}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^3} \sum_{i=1}^{I-k} C_{i,k}^{3/2} + 3\frac{f_k \sigma_k^2}{\sum_{i=1}^{I-k} C_{i,k}} + f_k^3 \tag{9}$$

For the second element of equation (11), we have:

$$E\left(\hat{f}_k C_{i,k+1}^2 \mid B_k\right) = \frac{1}{\sum_{j=1}^{I-k} C_{j,k}} E\left[\sum_{\substack{j=1\\ j\neq i}}^{I-k}\left(C_{j,k+1}C_{i,k+1}^2\right) + C_{i,k+1}^3 \mid B_k\right]$$

Due to the independence of $C_{i,k+1}$ and $C_{j,k+1}$, we have:

$$E\left(\hat{f}_k C_{i,k+1}^2 \mid B_k\right) = \frac{1}{\sum_{j=1}^{I-k} C_{j,k}}\left[\sum_{\substack{j=1\\ j\neq i}}^{I-k} E\left(C_{j,k+1} \mid B_k\right) E\left(C_{i,k+1}^2 \mid B_k\right) + E\left(C_{i,k+1}^3 \mid B_k\right)\right]$$

$$= \frac{1}{\sum_{j=1}^{I-k} C_{j,k}}\left[\left(C_{i,k}\sigma_k^2 + C_{i,k}^2 f_k^2\right) f_k \sum_{\substack{j=1\\ j\neq i}}^{I-k} C_{j,k} + Sk_k^3 C_{i,k}^{3/2} + 3 f_k C_{i,k}^2 \sigma_k^2 + C_{i,k}^3 f_k^3\right]$$

$$= \frac{1}{\sum_{j=1}^{I-k} C_{j,k}}\left[f_k C_{i,k}\sigma_k^2 \sum_{j=1}^{I-k} C_{j,k} + Sk_k^3 C_{i,k}^{3/2} + 2 f_k C_{i,k}^2 \sigma_k^2 + C_{i,k}^2 f_k^3 \sum_{j=1}^{I-k} C_{j,k}\right]$$

$$\Rightarrow E\left(\hat{f}_k C_{i,k+1}^2 \mid B_k\right) = f_k C_{i,k}\sigma_k^2 + \frac{Sk_k^3 C_{i,k}^{3/2}}{\sum_{j=1}^{I-k} C_{j,k}} + \frac{2 f_k C_{i,k}^2 \sigma_k^2}{\sum_{j=1}^{I-k} C_{j,k}} + C_{i,k}^2 f_k^3 \qquad (10)$$

Finally, for the third element of equation (7), we have:

$$E\left(\hat{f}_k^2 C_{i,k+1} \mid B_k\right) = \frac{1}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^2} E\left[C_{i,k+1}\left(\sum_{j=1}^{I-k} C_{j,k+1}\right)^2 \mid B_k\right]$$

$$= \frac{1}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^2} E\left[C_{i,k+1}\left(\sum_{j=1}^{I-k} C_{j,k+1}^2 + 2\sum_{n=1}^{I-k}\sum_{\substack{m=1\\ m\neq n}}^{n} C_{n,k+1}C_{m,k+1}\right) \mid B_k\right]$$

$$= \frac{1}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^2} E\left[C_{i,k+1}^3 + \sum_{\substack{j=1\\ j\neq i}}^{I-k}\left(C_{j,k+1}^2 C_{i,k+1}\right) + 2\sum_{\substack{n=1\\ n\neq i}}^{I-k}\sum_{\substack{m=1\\ m\neq n\\ m\neq i}}^{n}\left(C_{n,k+1}C_{m,k+1}C_{i,k+1}\right) + 2\sum_{\substack{n=1\\ n\neq i}}^{I-k}\left(C_{i,k+1}^2 C_{n,k+1}\right) \mid B_k\right]$$

Due to the independence of accident years (e.g. $C_{i,k+1}$ and $C_{j,k+1}$), we have:

$$E\left(\hat{f}_k^2 C_{i,k+1} \mid B_k\right) = \frac{1}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^2} \left[\begin{array}{l} E\left(C_{i,k+1}^3 \mid B_k\right) + \sum_{\substack{j=1 \\ j\neq i}}^{I-k} E\left(C_{j,k+1}^2 \mid B_k\right) E\left(C_{i,k+1} \mid B_k\right) \\ +2\sum_{\substack{n=1 \\ n\neq i}}^{I-k} \sum_{\substack{m=1 \\ m\neq n \\ m\neq i}}^{n} E\left(C_{n,k+1} \mid B_k\right) E\left(C_{m,k+1} \mid B_k\right) E\left(C_{i,k+1} \mid B_k\right) \\ +2\sum_{\substack{n=1 \\ n\neq i}}^{I-k} E\left(C_{i,k+1}^2 \mid B_k\right) E\left(C_{n,k+1} \mid B_k\right) \end{array} \right]$$

$$= \frac{1}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^2} \left[\begin{array}{l} Sk_k^3 C_{i,k}^{3/2} + 3 f_k C_{i,k}^2 \sigma_k^2 + C_{i,k}^3 f_k^3 + f_k C_{i,k} \sum_{\substack{j=1 \\ j\neq i}}^{I-k} \left(C_{j,k}\sigma_k^2 + C_{j,k}^2 f_k^2\right) \\ +2 f_k^3 \sum_{\substack{n=1 \\ n\neq i}}^{I-k} \sum_{\substack{m=1 \\ m\neq n \\ m\neq i}}^{n} C_{n,k} C_{m,k} C_{i,k} + 2\left(C_{i,k}\sigma_k^2 + C_{i,k}^2 f_k^2\right) f_k \sum_{\substack{n=1 \\ n\neq i}}^{I-k} C_{n,k} \end{array} \right]$$

Rearranging the terms, we have the equation:

$$E\left(\hat{f}_k^2 C_{i,k+1} \mid B_k\right) = \frac{1}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^2} \left[Sk_k^3 C_{i,k}^{3/2} + 3 f_k C_{i,k} \sigma_k^2 \sum_{n=1}^{I-k} C_{n,k} + C_{i,k} f_k^3 \left(\sum_{n=1}^{I-k} C_{n,k}\right)^2 \right] \qquad (11)$$

Introducing equation (8), (9), (10) and (11) in equation (7), we finally get:

$$\begin{aligned}
(Eq) = & \sum_{i=1}^{I-k} \left[\frac{1}{C_{i,k}^{3/2}} \left(Sk_k^3 C_{i,k}^{3/2} + 3 f_k C_{i,k}^2 \sigma_k^2 + C_{i,k}^3 f_k^3 \right) \right] \\
& -3\sum_{i=1}^{I-k} \left[\frac{1}{\sqrt{C_{i,k}}} \left(f_k C_{i,k} \sigma_k^2 + \frac{Sk_k^3 C_{i,k}^{3/2}}{\sum_{j=1}^{I-k} C_{j,k}} + \frac{2 f_k C_{i,k}^2 \sigma_k^2}{\sum_{j=1}^{I-k} C_{j,k}} + C_{i,k}^2 f_k^3 \right) \right] \\
& +3\sum_{i=1}^{I-k} \left[\sqrt{C_{i,k}} \left(\frac{1}{\left(\sum_{j=1}^{I-k} C_{j,k}\right)^2} \left[Sk_k^3 C_{i,k}^{3/2} + 3 f_k C_{i,k} \sigma_k^2 \sum_{n=1}^{I-k} C_{n,k} + C_{i,k} f_k^3 \left(\sum_{n=1}^{I-k} C_{n,k}\right)^2 \right] \right) \right] \\
& - \sum_{i=1}^{I-k} C_{i,k}^{3/2} \left(\frac{Sk_k^3}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^3} \sum_{i=1}^{I-k} C_{i,k}^{3/2} + 3\frac{f_k \sigma_k^2}{\sum_{i=1}^{I-k} C_{i,k}} + f_k^3 \right)
\end{aligned}$$

Most of the terms of the above equation eliminate each other and we finally get:

$$(Eq) = \left(I-k\right)Sk_k^3 - Sk_k^3 \frac{\left(\sum_{i=1}^{I-k} C_{i,k}^{3/2}\right)^2}{\left(\sum_{i=1}^{I-k} C_{i,k}\right)^3}$$

The proof of the unbiasedness is completed on noting that:

$$E\left(\hat{S}k_k^3\right) = E\left(E\left(\hat{S}k_k^3 \mid B_k\right)\right) = Sk_k^3$$

APPENDIX 2 – Definition and properties of Gaussian copulas

Definition of a copula function

An n-dimensional copula (See Sklar (1959)) is a multivariate distribution function, C, with uniform distributed margins in [0,1] and the following properties:

1. $C:[0,1]^n \to [0,1]$
2. C is grounded and n-increasing
3. C has margins C_i which satisfy $C_i(u) = C(1,...,1,u,1,...,1) = u$ for all $u \in [0,1]$.

It is obvious from the above definition, that, if F_1, …, F_n are univariate distribution functions, $C(F_1(x_1),\dots, F_n(x_n))$ is a multivariate distribution function with margins F_1, …, F_n. Copula functions are a useful tool to construct and simulate multivariate distributions.

Sklar's theorem: Let F be an n-dimensional distribution function with continuous margins F_1, …, F_n. Then it has the following unique copula representation:

$$F(x_1,\dots, x_n) = C(F_1(x_1),\dots, F_n(x_n)) \tag{2}$$

For the proof of the theorem, see Sklar (1996).

Corollary: Let F be an n-dimensional distribution function with continuous margins F_1, …, F_n.and copula C satisfying (2). Then, for any $u=(u_1,\dots, u_n)$ in $[0,1]^n$:

$$C(u_1,...,u_n) = F\left(F_1^{-1}(u_1),...,F_n^{-1}(u_n)\right)$$

where F_i^{-1} is the generalized inverse of F_i.

Gaussian copula

If X has the stochastic representation:

$$X \overset{d}{=} \mu + AZ \tag{12}$$

where $Z_1,...,Z_n \sim \mathrm{N}(0,1)$ are independent.

Then X has an n-variate Gaussian distribution with mean vector μ and covariance matrix $\Sigma = AA^T$.

The copula of the n-variate normal distribution with linear correlation matrix Σ is:

$$C_\Sigma(u) = \Phi_\Sigma^n\left(\Phi^{-1}(u_1),...,\Phi^{-1}(u_n)\right)$$

where Φ_Σ^n denotes the joint distribution function of the n-variate standard normal distribution function with linear correlation matrix Σ, and Φ^{-1} denotes the inverse of the distribution function of the univariate standard normal distribution.

It can be shown that the Gaussian copula has no upper or lower tail dependence (see Embrechts, Lindskog and McNeil (2001)).

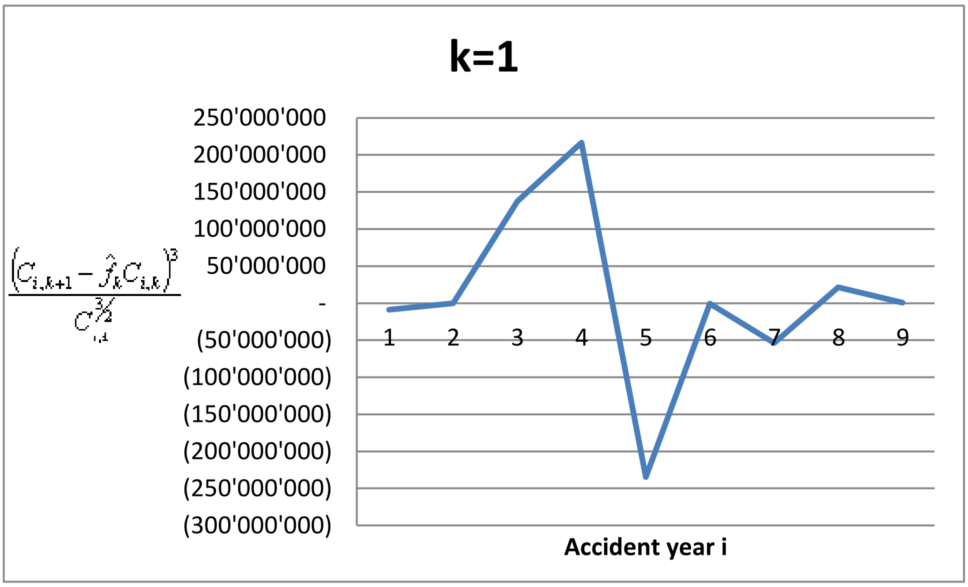
To simulate random variates from the Gaussian copula, C_Σ , we can use the following algorithm based on equation (12):

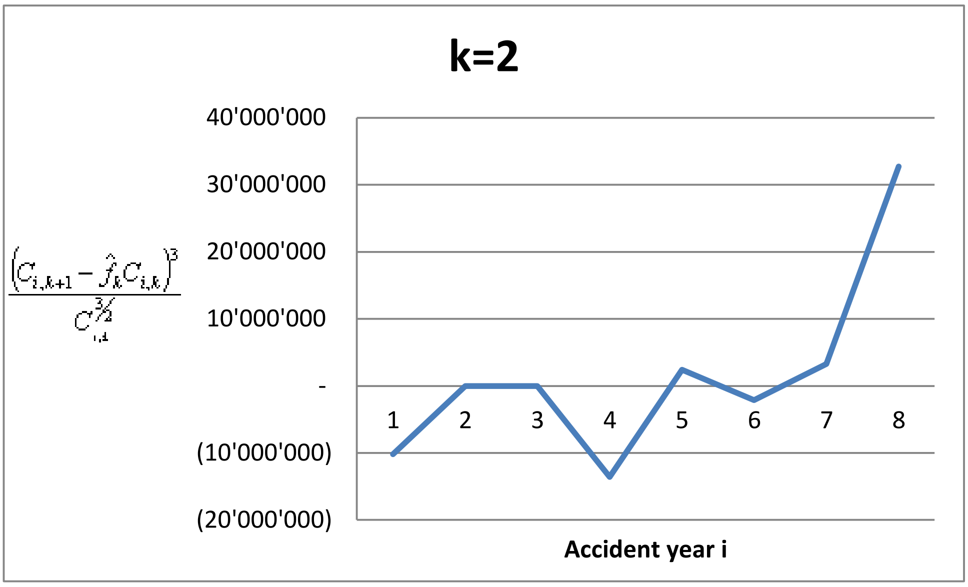
- find the Cholesky decomposition, A, of Σ so that $\Sigma = A^t A$
- simulate n independent random variates $z = \begin{pmatrix} z_1 \\ ... \\ z_n \end{pmatrix}$ from the standard normal distribution
- set the vector y = Az
- determine the components $u_i = \Phi(x_i), i = 1,...,n$
- the resulting vector is: $u = \begin{pmatrix} u_1 \\ ... \\ u_n \end{pmatrix} \sim C_\Sigma$.

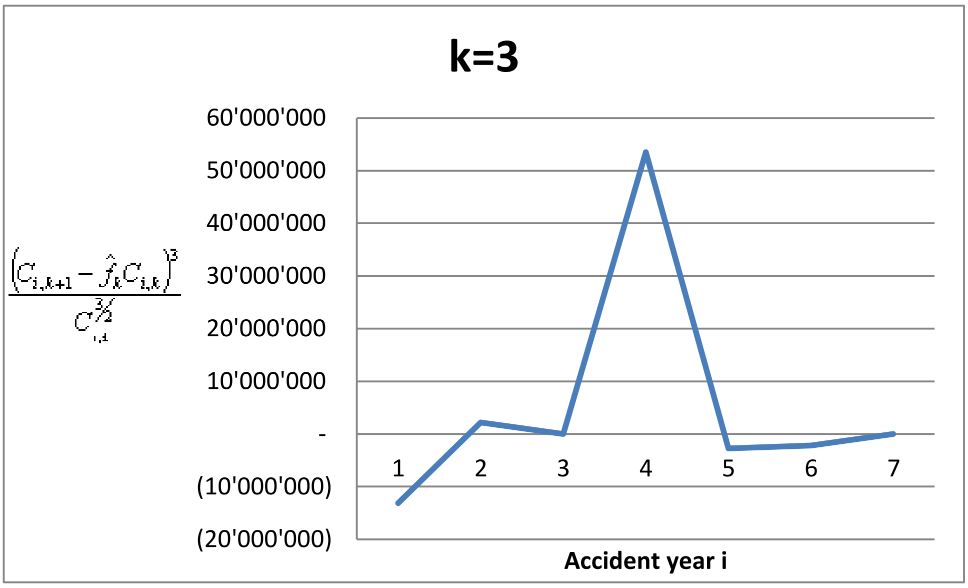
APPENDIX 3 – Graphs

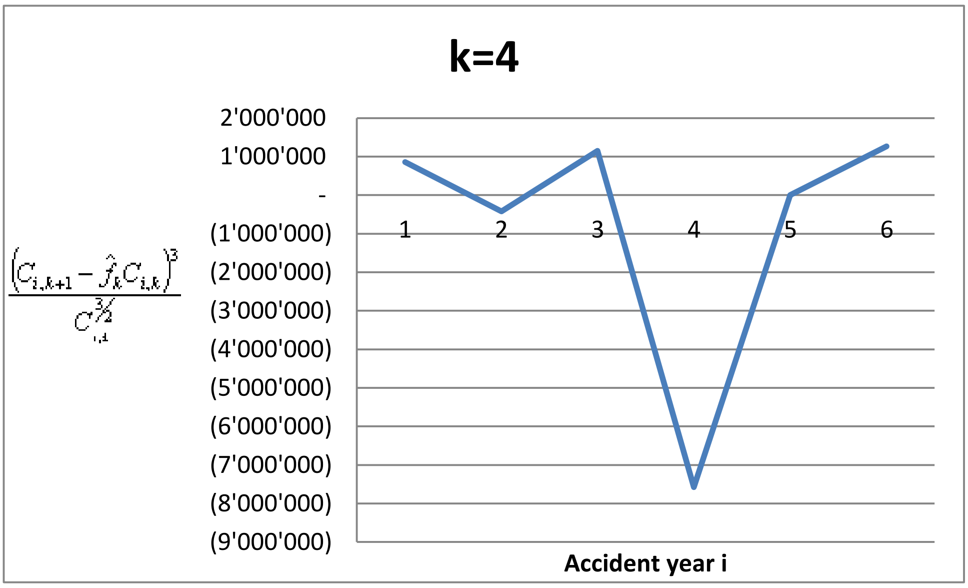
In this appendix, graphs of $\frac{\left(C_{i,k+1}-\hat{f}_k C_{i,k}\right)^3}{C_{i,k}^{3/2}}$ against $C_{i,k}^{3/2}$ are provided for each numerical example given in section 5 and for each development year k. Overall, there does not seem to be any tendencies on the accident year i and, as a consequence, the methodology presented in this paper should be applicable.

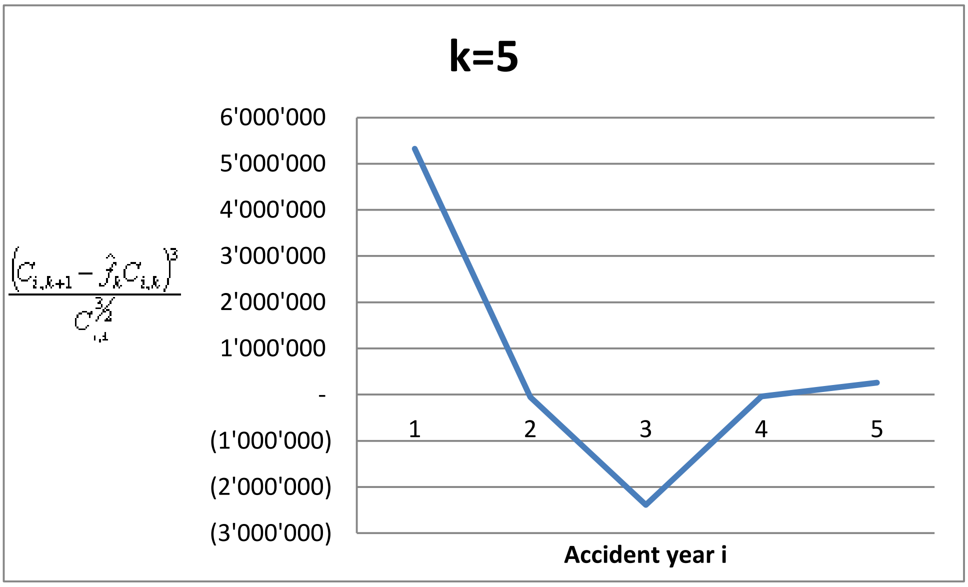
Example 1



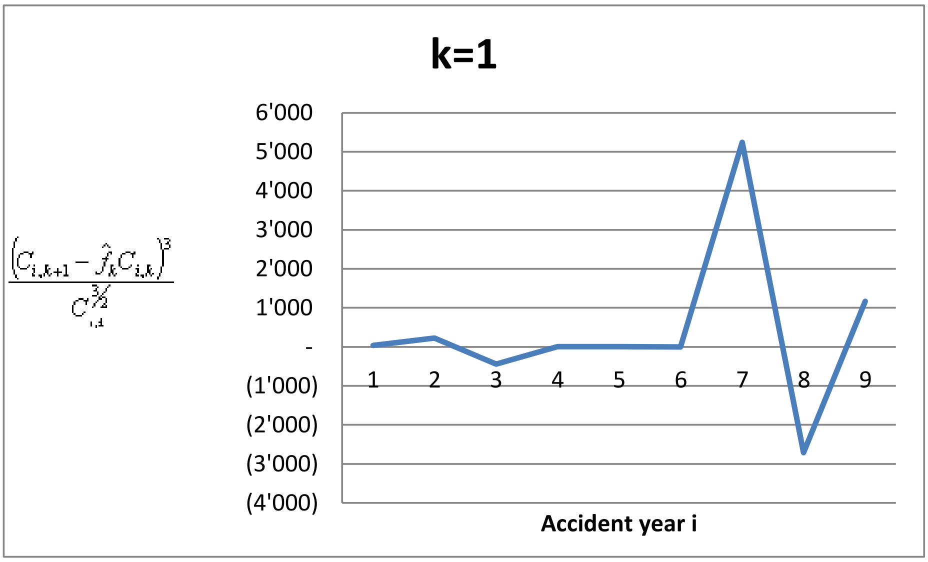



k=3
60'000'000
50'000'000
40'000'000
30'000'000
20'000'000
10'000'000
-
(10'000'000)
(20'000'000)
1
2
3
4
5
6
7
Accident year i

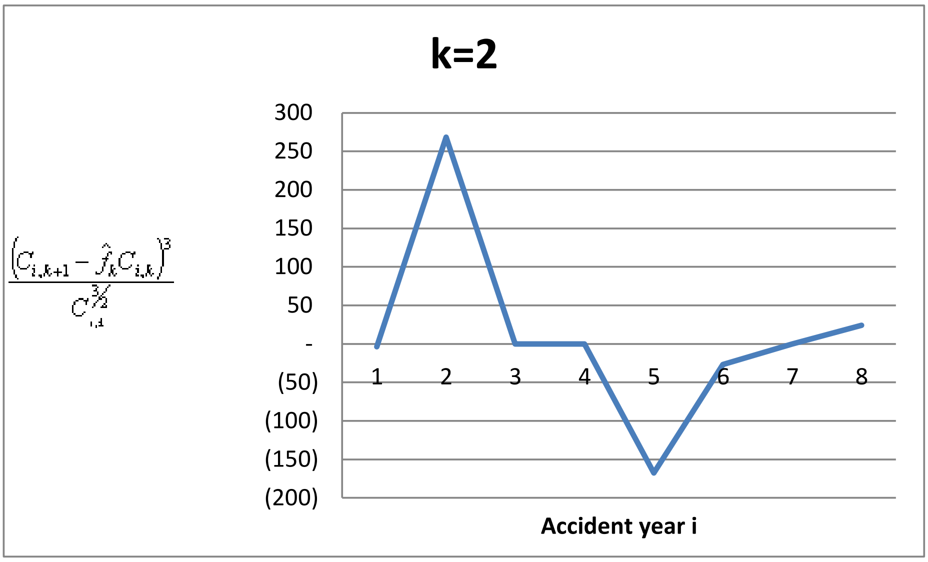

k=4
2'000'000
1'000'000
-
(1'000'000)
(2'000'000)
(3'000'000)
(4'000'000)
(5'000'000)
(6'000'000)
(7'000'000)
(8'000'000)
(9'000'000)
1
2
3
4
5
6
Accident year i


k=5
6'000'000
5'000'000
4'000'000
3'000'000
2'000'000
1'000'000
-
(1'000'000)
(2'000'000)
(3'000'000)
1
2
3
4
5
Accident year i

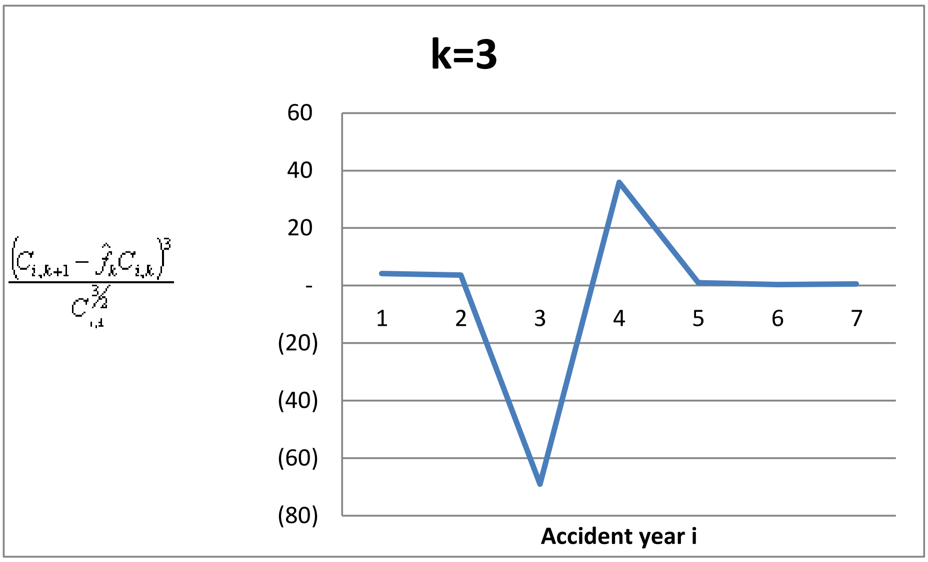
Example 2



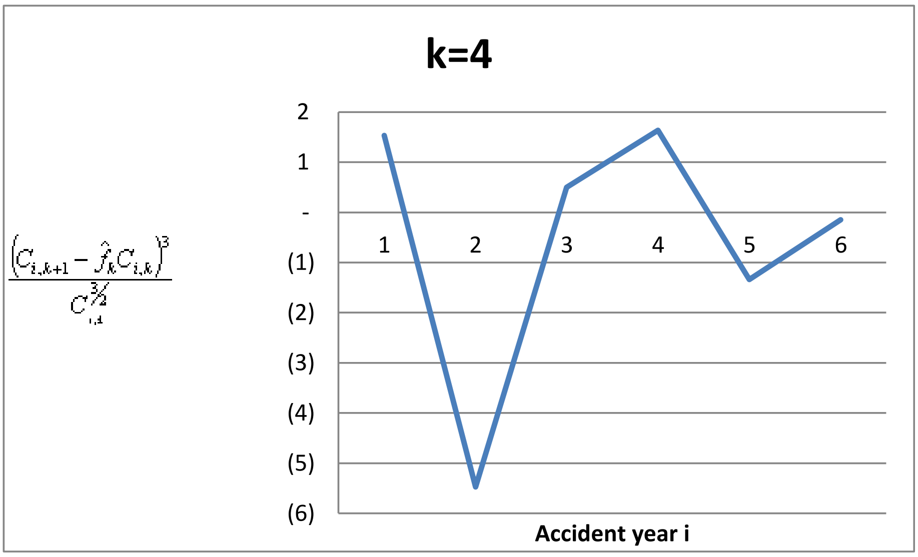
k=1
$\frac{(C_{i,k+1}-\hat{f}_k C_{i,k})^3}{C_{i,k}^{3/2}}$
6'000
5'000
4'000
3'000
2'000
1'000
-
(1'000)
(2'000)
(3'000)
(4'000)
1 2 3 4 5 6 7 8 9
Accident year i



k=2
$\frac{(C_{i,k+1}-\hat{f}_k C_{i,k})^3}{C_{i,k}^{3/2}}$
300
250
200
150
100
50
-
(50)
(100)
(150)
(200)
1 2 3 4 5 6 7 8
Accident year i



k=3
$\frac{(C_{i,k+1}-\hat{f}_k C_{i,k})^3}{C_{i,k}^{3/2}}$
60
40
20
-
(20)
(40)
(60)
(80)
1 2 3 4 5 6 7
Accident year i


k=4
$\frac{(C_{i,k+1} - \hat{f}_k C_{i,k})^3}{C_{i,k}^{3/2}}$
2
1
-
(1)
(2)
(3)
(4)
(5)
(6)
1
2
3
4
5
6
Accident year i