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Dynamic optimal investment in Markov-modulated Lévy markets with risk of default and general utility function

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A general stochastic differential equation

$$
\begin{aligned}
\mathrm{d}U_t &= a(U_t, Y_t, \sigma_t)\mathrm{d}t + b(U_t, Y_t, \sigma_t)\mathrm{d}W_t + \\
&\quad + \int_{\mathbb{R}^k} \gamma(U_{t-}, Y_{t-}, \sigma_{t-}, z) N(\mathrm{d}t, \mathrm{d}z) \qquad (1) \\
U_0 &= u
\end{aligned}
$$

where a, b and γ are known real valued functions, W_t is a standard Wiener process and $N(\mathrm{d}t, \mathrm{d}z)$ is a compensated Poisson random measure. See Bertoin (1998) for more details on Lévy processes. Observe that the process defined by (1) is time-homogeneous, therefore its evolution is invariant under time shifts.

The process σ_t is the control assumed to be adapted and càdlàg. The process Y_t is a Markov process with finite state space $\{\delta_1, \ldots, \delta_m\}$ and intensity matrix $Q = \{q_{ij}\}$ that modulates the coefficients of (1).

An example

The risk process considered follows

$$dR_t = c\,dt + \rho\,dW_t - dX_t, \qquad R_0 = u \tag{19}$$

where X_t is a compound Poisson process with intensity $\lambda = \frac{1}{3}$ and lognormal claim size distribution $\mathcal{LN}(1,2)$. This process represents claims collected at a constant rate $c = 3$ perturbed by a diffusion with volatility $\rho^2 = 0.25$ that can be interpreted as aggregate small claims and claims collection accruals. The Poisson process then represents catastrophic claims (with average occurrence once every 3 periods) with lognormal (heavy-tail) severity distribution. The investment opportunities will be represented by a riskless asset $dS_t^{(1)} = r\,dt$ and a risky asset $dS_t^{(2)} = \nu\,dt + \xi\,dW_t$. The proportion invested into a risky asset will be denoted as π. No short-selling is allowed, therefore $\pi \in [0,1]$.

An example

Altogether, the reserve process, including investment, can be written as

$$\mathrm{d}U_t = [c + (r + \pi(\nu - r))U_t]\,\mathrm{d}t + \sqrt{\rho^2 + \pi^2\xi^2 U_t^2}\,\mathrm{d}W_t - \mathrm{d}X_t \qquad U_0 = u \quad (20)$$

This implies the following linear relationship between the volatility $\sigma = \pi\xi$ and expected return on investment $\mu\sigma$

$$\mu(\sigma) = r + \pi(\nu - r) = r + \sigma\frac{\nu - r}{\xi}. \quad (21)$$

A complex optimization problem

The stochastic control is considered in a fixed horizon T or until the process U_t exits a region $S \subseteq \mathbb{R}$. The performance criterion v to be maximised is

$$v^\sigma(t, T, u, \delta_i) \equiv \mathrm{E}\left[P(U_T, Y_T)\mathbb{I}_{\{\tau \geq T\}} + \right. \\ \left. + L(U_\tau, Y_\tau)\mathbb{I}_{\{t<\tau<T\}} | U_t = u, Y_t = \delta_i\right] \tag{2}$$

where $\tau = \inf\{s \geq t : U_s \notin S\}$ is the exit time of the process U_s from the region S. Functions P and L are arbitrary but continuous and represent the utility realised upon termination of the horizon and at exit of the controlled process from the region S.

Let us denote $J(t, T, u, \delta_i)$ the optimal value of the maximisation problem

$$J(t, T, u, \delta_i) \equiv \max_{\sigma \in \Pi} v^\sigma(t, T, u, \delta_i). \tag{3}$$

where the set Π contains all the admissible controls, that is such σ_t adapted for which a strong solution to the equation (1) exists and is unique. The attention will be restricted to controls of the form $\sigma_t = \sigma(U_{t-}, Y_{t-})$ also called Markov controls. Øksendal (2003, Th. 11.2.3.) gives fairly week sufficient conditions under which the optimal value of the problem restricted to Markov controls equals the optimal value of the problem with arbitrary adapted control. Therefore narrowing the control space Π to Markov controls is not too restrictive.

A complex optimization problem

Solving (3) directly is not feasible since an explicit expression for $v^\sigma(t,T,u,\delta_i)$ is not available in the most general case. Following the dynamic programming approach, one can write the Hamilton-Jacobi-Bellman equation that characterises the value function $J(t,T,u,\delta_i)$

$$\sup_{\sigma\in\Pi}\left\{\mathcal{A}^\sigma J(t,T,u,\delta_i)+\frac{\partial J(t,T,u,\delta_i)}{\partial t}\right\}=0 \tag{4}$$

Moreover, if σ^* is such that

$$\mathcal{A}^{\sigma^*} J(t,T,u,\delta_i)+\frac{\partial J(t,T,u,\delta_i)}{\partial t}=0 \tag{6}$$

then σ^* is the optimal control for (3) and $J(t,T,u,\delta_i)=v^{\sigma^*}(t,T,u,\delta_i)$.

In this case, the optimal control can be found as the maximiser of the expression under the supremum in (4) whence the following must hold

$$\frac{\partial}{\partial\sigma}\mathcal{A}^{\sigma^*} J(t,T,u,\delta_i)=0. \tag{7}$$

A complex optimization problem

The HJB equation (4) together with optimality condition (7), in the view of (5), form a system of non-linear second order partial integro-differential equations. The solution of such system is usually a very difficult task even using numerical procedures. Analytic solutions are seldom found due to:

- Optimal control condition (7) may not yield an explicit value for σ in a closed formula.
- The admissible controls $\sigma \in \Pi$ may not be continuous and only finite or infinite countable.
- The PDE contained in curly brackets in (4) lacks of an explicit solution in most of the practical cases, and most important
- even though the previous assumptions hold, the optimal control usually depend on the value function $J(t, T, u, \delta_i)$

The erlangian approach

The maximisation problem (3) will be now considered in a hypothetical backwards $\mathrm{Er}\left(n/\left(T-t\right),n\right)$ random time H_n, independent of (U_s, Y_s, W_s). Let Υ_n^σ be the performance criterion to be maximised in the mentioned random backwards horizon H_n with terminal time T

$$\begin{aligned}\Upsilon_n^\sigma(H_n, T, u, \delta_i) \equiv \mathrm{E}_{H_n}\left[v^\sigma\left(T-H_n, T, u, \delta_i\right)\right] \\ = \mathrm{E}_{H_n}\left[\mathrm{E}\left[P(U_T, Y_T)\mathbb{I}_{\{\tau\geq T\}}+\right.\right. \\ \left.\left.+L(U_\tau, Y_\tau)\mathbb{I}_{\{T-H_n<\tau<T\}}\right|U_{T-H_n}=u, Y_{T-H_n}=\delta_i\right]\right]\end{aligned}$$

where now $\tau=\inf\left\{s\geq T-H_n : U_s\notin S\right\}$ is the exit time of the process U_s from the region S.

The optimisation problem is now defined as

$$J_n(H_n, T, u, \delta_i) \equiv \max_{\sigma\in\Pi}\Upsilon_n^\sigma(H_n, T, u, \delta_i) \tag{8}$$

The erlangian approach

This results in elimination of the time dependence of the optimisation problem. Now, moreover, the control that in principle is an adapted process evolving in time, will be restricted to a piecewise constant process (constant on each exponential horizon composing H_n). By intuition, since the length of each exponential interval is infinitesimal with probability1 as n increases, the optimisation on restricted set of controls will converge to the unrestricted one, therefore the convergence of the procedure outlined earlier is not compromised. Theorem 3 in this section proves this idea formally.

We will introduce some necessary notation. Let J_1 be the value function of the problem in exponential horizon restricted to a piecewise constant control

$$\overline{J}_1(H_1, T, u, \delta_i) \equiv \max_{\sigma} \Upsilon^{\sigma}(H_1, T, u, \delta_i) \tag{14}$$

The main results

Assuming that controls are restricted to be constant on each exponential interval composing the Erlangian horizon H_n, the value function of the restricted problem is denoted $\overline{J}_n$. The set of all piecewise constant controls is $\overline{\Pi}$ and

$$\overline{J}_n(H_n, T, u, \delta_i) \equiv \max_{\sigma \in \overline{\Pi}} \Upsilon_n^{\sigma}(H_n, T, u, \delta_i). \tag{15}$$

Theorem 3. *Let $J(t, T, u, y)$ be the value function (3) and $\overline{J}_n(H_n, T, u, \delta_i)$ the value function (15) then for each $u \geq 0$ and δ_i*

$$\lim_{n \to \infty} \overline{J}_n(H_n, T, u, \delta_i) = J(t, T, u, \delta_i)$$

Corollary 1. *Let P be a reward function, u the initial condition, $\alpha > 0$ a real parameter. For every natural $k \geq 2$*

$$\overline{J}_k(H_k, T, u, \delta_i, P) = \overline{J}_1(H_1, T, u, \delta_i, \overline{J}_{k-1}) \tag{17}$$

...just a sequence of standard optimizations suffices

The value function $\overline{J}_1$ is the basic building block of the method. Since in each exponential horizon the control is constant, it can be treated as a parameter and $\Upsilon_1^\sigma(H_1, T, u, \delta_i)$ can be obtained from the Laplace-Carson transform of the usual Fokker-Planck equation (see i.e. Risken (1996))

$$\mathcal{A}^\sigma \Upsilon^\sigma (H_1, T, u, \delta_i) + \alpha \Upsilon^\sigma (H_1, T, u, \delta_i) - \alpha P(u, \delta_i) = 0. \tag{18}$$

Posterior maximisation with respect to σ yields $\overline{J}_1$. Notice that the maximisation is a standard optimisation problem on real numbers.

The simplicity of the equation (18) to be solved, compared to system (6) and (7), is the main advantage of the method that together with Corollary (17) provides a semi-analytic treatment of the stochastic control problem presented.

An example...

$$v^{\sigma}(T,u) = \mathrm{E}\left[\mathbb{I}_{\{\tau \geq T\}} | U_0 = u\right] = \mathrm{P}\left[\tau \geq T | U_0 = u\right] \equiv \varphi(u,T) \tag{22}$$

what represents the survival probability. The optimisation problem (3) then turns to maximisation of survival probability in a fixed horizon T. Similar problems have been treated in Hipp and Plum (2003) and others but no closed form solution exist. Following the development presented above, in order to be able to apply the iterative scheme from Theorem 2, the fixed horizon T will be approximated by a series of n exponential horizons with parameter $\frac{n}{T}$

$$\varphi^{*}(u,\alpha) = \int_{0}^{\infty} \varphi(u,T)\alpha \mathrm{e}^{-\alpha T} \mathrm{d}t. \tag{23}$$

An example...

function Υ for a constant π satisfies satisfies the following integro-differential equation

$$\frac{1}{2}(\rho^2+\pi^2\xi^2u^2)\frac{\partial^2}{\partial u^2}\Upsilon+(c+(r+\pi(\nu-r))u)\frac{\partial}{\partial u}\Upsilon \\ -(\lambda+\alpha)\Upsilon+\lambda\int_0^u\Upsilon(\alpha,u-x)f(x)\mathrm{d}x+\alpha J_{i-1}(u)=0. \quad (25)$$

as derived in Diko and Usábel (2011) an approximation method by chebyshev polynomials is used to calculate the solution to this problem. Since feasible strategies are bounded it is possible to evaluate Υ for a grid of possible values of $\pi\in[0,1]$ and take the maximum value as an approximation to the solution of (24). In this example we took equidistant grid of granularity 0.1. The table 1 shows the results of approximated value function $J(u,T)$ for various values of initial reserves u and number of exponential intervals n that approximate the fixed horizon $T=10$. The convergence is achieved to up to 3 decimal places for as few as 100 intervals.

An example…

n – number of intervals

u		1	2	5	10	20	50	100
	0.1	0.354370	0.306438	0.287529	0.288241	0.291537	0.295415	0.297638
	0.5	0.413865	0.361477	0.341409	0.341409	0.346224	0.349630	0.350752
	1	0.427487	0.377567	0.359446	0.361308	0.365365	0.369168	0.370406
	2	0.456898	0.412100	0.397678	0.400614	0.405446	0.410055	0.411589
	5	0.570571	0.542250	0.536840	0.541189	0.547234	0.554050	0.556663
	10	0.886121	0.882775	0.883860	0.885951	0.888517	0.891156	0.891655
	15	0.999024	0.998997	0.999008	0.999027	0.999049	0.999072	0.999074

Thank you…

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