ASTIN, AFIR/ERM and IACA Colloquia
Innovation & Invention

23-27 August 2015 | Amora Hotel Jamison Sydney

Actuaries Institute

GMxB forward looking risk measures

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Background and Motivation

- Guaranteed Minimum Benefit products (GMxB's) are popular in North America and have a growing presence in Europe and Asia (incl Australia).
- Monte Carlo methods are critical to the valuation and hedging of liabilities associated with such products.
- Great scope and possibilities for application of recent advances in Monte Carlo techniques to these problems. These advances allow us to calculate all sensitivities with same Monte Carlo scenarios used for the Liability and with high precision.
- Current state-of-the-art hedging programs are centered around point valuation of liabilities, and bump and re-price approach for sensitivities (greeks). This method is slow & provides estimates of Gamma that have high standard errors.
- Current methods fall short of a "full solution". Briefly, a full solution means:

Given a particular hedging strategy what is the ***DISTRIBUTION*** of the hedging P&L (liability minus hedge assets) at some point of interest in the future.

Background and Motivation

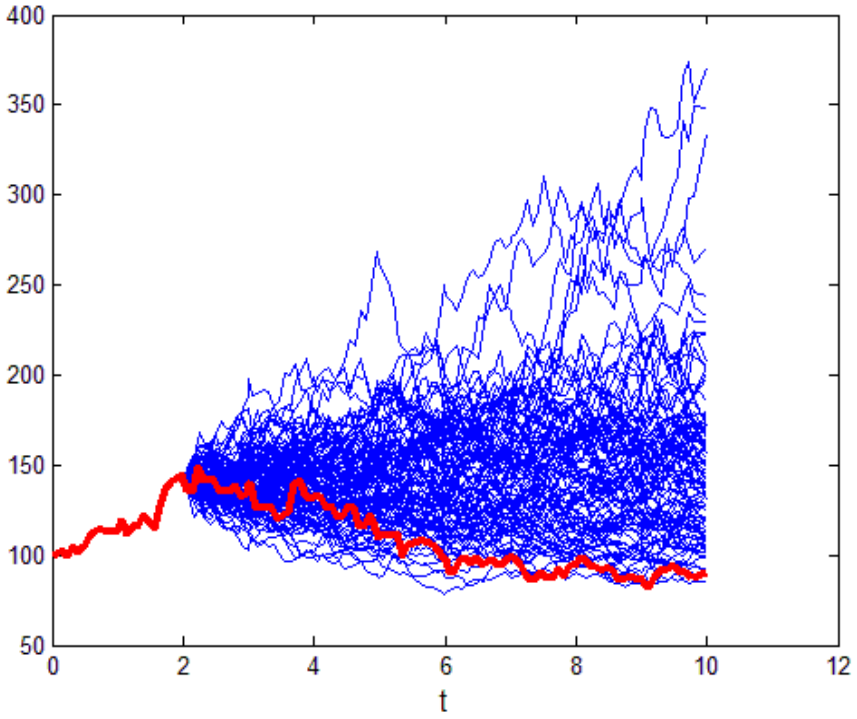
- Full solution requires:
  - stochastic on stochastic simulations that project liability characteristics into the future with a high degree of accuracy
  - A platform for simulating a hedging strategy applied to liabilities projected into the future

- Major contributions:
  - Development of methods to compute delta and gamma simultaneously with liability. No bump and re-price.
  - Gamma calculation is vastly more accurate than bump and re-price method.
  - Tools that use these techniques to simulate the evolution of key risk measures (gamma) and performance of hedging strategies.

Hedge P&L distribution calculation

Basic Concepts:

- **Outer Loop;** model of real world, or based on historical data. This is where we perform hedging transactions and make or lose real money.
- **Inner Loop;** assumed model for evolution of underlying assets (eg a vector GBM) and interest rates. This is the model used to price guarantee and evaluate sensitivities.



- At each point of outer loop (red) where we want to rebalance hedges we will need to compute liability and greeks.
- Inner loop Monte Carlo scenarios (blue) are produced at each point in the outer loop
- Liability and various greeks are computed using these scenarios.
- Rebalancing hedges less frequently than daily introduces significant errors in simulation of the P&L.

Inner Loop Computations

Suppose we have sold a guarantee over a basket of assets. The basket has **d** components.

To simulate a hedge P&L that rebalances daily over 10 years will require (**d(d+1)* 3600**) liability calculations per contract.

- What exactly is required to be computed from the inner loop at each point of the outer loop?
  - The Liability
  - The Delta – d components requires each element of the underlying to be perturbed and liability to be re-computed generally uses 2 additional liability calculations.
  - The Gamma – full cross gamma matrix is d x d and symmetric, d(d+1)/2 components to be computed. Each component requires 2 additional liability calculations.
  - Other sensitivities (rho).

Outer Loop Computations

- With sensitivities computed; hedging can be performed in the outer loop
- Delta hedging; use futures to match the first order equity sensitivities
- Rho hedging; use IR Swaps
- Rho Convexity: use interest rate swaptions where rho hedge IR swap convexity do not fully match
- Gamma Hedging; use a put option over a similar basket of assets.

If we delta hedge Gamma losses along a single outer loop path are given by:

$$\text{P \& L} = \sum_{i=1}^{M} \left[\sigma_{realised}(t_i)^T \Gamma_S(t_i) \sigma_{realised}(t_i) - \sigma_{assumed}^T \Gamma_S(t_i) \sigma_{assumed} \right] (t_{i+1} - t_i)$$

$$\sigma_{realised}(t_i) = \left[\ln \frac{S_1(t_{i+1})}{S_1(t_i)} \quad \ln \frac{S_1(t_{i+1})}{S_1(t_i)} \quad \cdots \quad \ln \frac{S_d(t_{i+1})}{S_d(t_i)} \right]^T$$

$\sigma_{assumed}$ – inner-loop volatility.

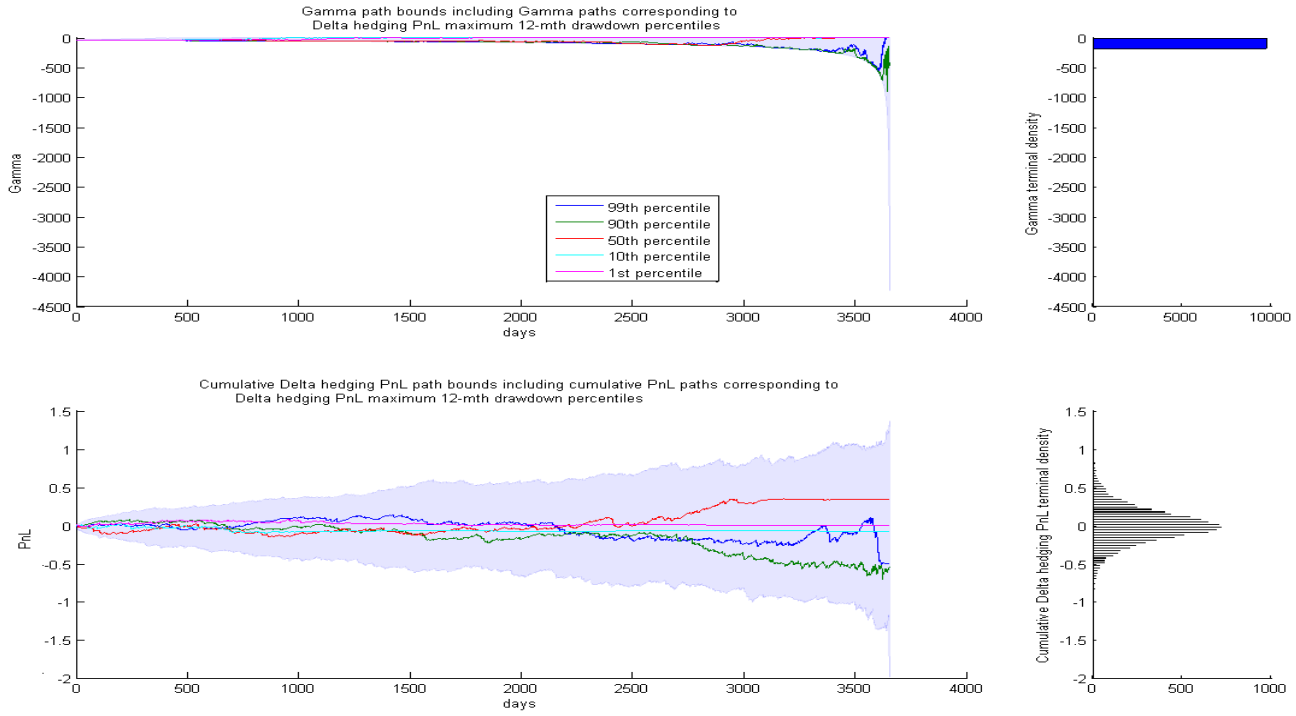
$\Gamma_S(t_i)$ – Dollar Gamma of the liability at t_i.

If we delta hedge we expected Gamma losses across all outer loop paths is given by:

$$\text{Average P\&L} = \frac{1}{\#\text{Paths}} \sum_{j=1}^{\#\,paths} \text{P\&L of path } j$$

Forward Looking Risk Measure

Losses are dependent on both volatility and gamma and so risk is path dependent i.e. final asset price in the outer loop does not tell us much about the potential losses we could experience.



Possible forward looking risk measures:

- 99.5% ile 12 month drawdown
- Expected future losses if volatility is higher than expected

Problem Statement

- Given a set of d underlying assets represented by vector GBM:

$$\mathbf{S}(t) = \left(S_1(t), S_2(t), \ldots, S_d(t)\right) = \mathbf{S}(0) \circ \exp\left\{\left(r\mathbf{1} - \frac{1}{2}\Sigma^2\mathbf{1}\right)t + \sqrt{t}\mathbf{C}^{1/2}\mathbf{Z}\right\}$$

- Standard definitions:

r – riskless rate

$\mathbf{C}^{1/2}$ – square root covariance matrix

$\Sigma = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_d\}$ – volatilities

$\mathbf{Z} \sim N(\mathbf{0}_d, \mathbf{I}_d)$ – standard d-dimensional multivariate normal

$\mathbf{1}$ – vector of 1's

- Basket:

$$S_B(t) = \sum_{i=1}^{d} w_i S_i(t) = \mathbf{w}^T \mathbf{S}(t)$$

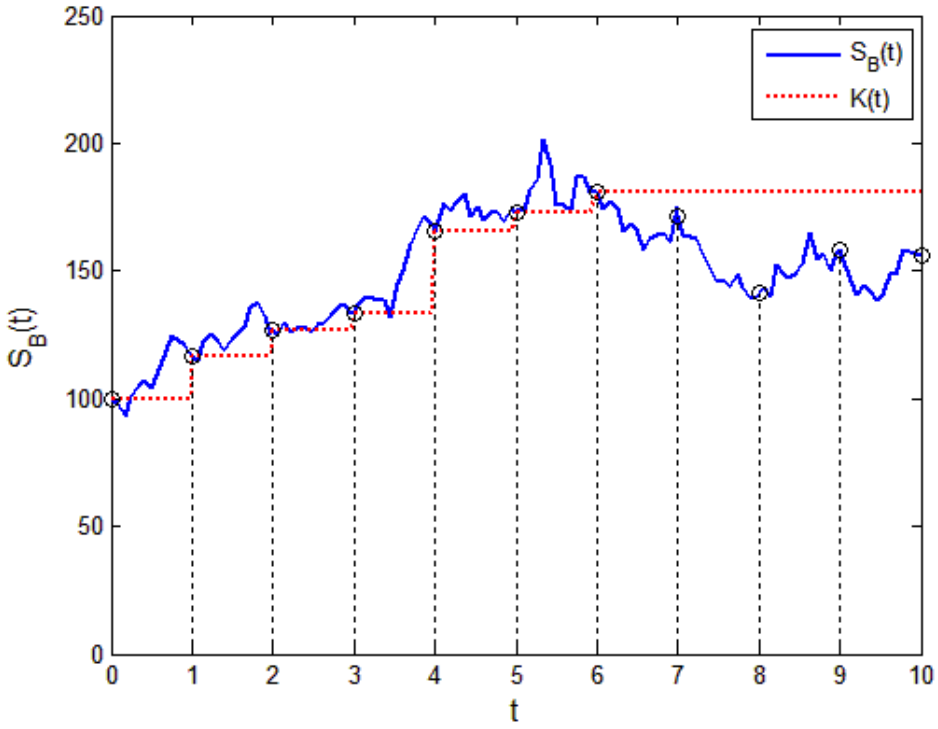
Problem Statement

- Payoff at maturity T = 10:

$$C(T) = \max\{K(T) - S_B(T), 0\} \qquad C(T) - \text{claim amount}$$

$$K(T) = \max\{K(0), S_B(t_1), S_B(t_2), \ldots, S_B(t_n)\} \qquad K(T) - \text{ratcheting strike}$$

- Note: This is not the exact payoff or liability – but the important part for computation of Greeks.



$$C(10) = K(10) - S_B(10)$$
$$= S_B(6) - S_B(10)$$

Fast Computation of Liability Greeks

- Joint work with Prof. Mark Joshi (University of Melbourne)
- Various approaches; pathwise, likelihood ratio, proxy sampling techniques.
- Interesting links between all approaches.
- Achievements:
  - For Delta: pathwise method allows computation of all **d** components of delta in same pass as liability
    - Equivalent to bump and re-price in the limit. Gains in computational efficiency.
  - For Gamma: Optimal Partial Proxy (OPP) method computes full Gamma matrix together with the liability/delta AND overcomes the instability of bump and re-price. Gains in both computational efficiency and accuracy.

Fast Computation of Delta

- Monte Carlo estimate for liability:

$$\hat{C}(\mathbf{S}(0), t=0) = \exp(-rT)\frac{1}{N}\sum_{n=1}^{N} C^n(T) \qquad C^n(T) - \text{payoff for individual path}$$

- Computes an estimate of

$$E_{RN}\{C(\mathbf{S}(0),T)\} = \exp\{-rT\}\int C(\mathbf{S}(0), \mathbf{Z}(t_1), \mathbf{Z}(t_2), \ldots, \mathbf{Z}(t_n), \mathbf{Z}(T))\phi(\mathbf{Z}(t_2), \ldots, \mathbf{Z}(t_n))d\mathbf{Z}(t_1)d\mathbf{Z}(t_2)\ldots d\mathbf{Z}(t_n)d\mathbf{Z}(T)$$

- Delta obtained by computing $\frac{\partial}{\partial S_i(0)}\hat{C}(\mathbf{S}(0), t=0), \quad i=1,\ldots,d$

$$\frac{\partial \hat{C}}{\partial S_i(0)} \approx \frac{1}{2h}\left(\hat{C}(S_i(0)+h, 0) - \hat{C}(S_i(0)-h, 0)\right) \qquad \text{(bump and re-price)}$$

$$\frac{\partial}{\partial S_i(0)}\hat{C}(\mathbf{S}(0), t=0) = \exp(-rT)\frac{1}{N}\sum_{i=1}^{N}\frac{\partial}{\partial S_i(0)}C^i(T)$$

- Is this allowed?

Fast Computation of Delta

- Possible for computing Delta as claim C satisfies the conditions.
- Pathwise estimate: analytical expression for $\frac{\partial C^n}{\partial S_i(0)}$ for each path.
- Bump and re-price converges to pathwise in the limit as h goes to 0.
- BUT in pathwise method: sensitivity computed for each path when liability is being evaluated – computational savings.

Interchange of differentiation and summation allowed when payoff is Lipschitz continuous in variable of interest leads to a pathwise estimator for delta of C.

$$\frac{\partial C^n(T)}{\partial S_j(0)} = \left[w_j \frac{S_j(t_*)}{S_j(0)} I_{[K(T) \neq K(0)]} - w_j \frac{S_j(T)}{S_j(0)} \right] I_{[S_B(T) < K(T)]}$$

Fast Computation of Gamma

- Can same trick be applied to Gamma?
- NO. The Delta cannot be differentiated for each path to get gamma. Delta function is discontinuous.
- Possible approaches: Likelihood Ratio, Proxy Simulation Schemes (OPP)

Fast Computation of Gamma – Likelihood Ratio

- Pathwise method relies on well behaved payoff: recall,
  - perturb parameter recompute payoff
  - Evaluate sensitivity for finite perturbation
  - Take 0 limit.

- Taking the limit as bump size goes to 0, needs well behaved payoff.
- If payoff is not nice (e.g. delta), what about the density function?
- Likelihood Ratio: View the parameters of interest **S**(0), as a parameter the of density rather than the path; then differentiate density with respect to **S**(0).

$$\frac{\partial}{\partial S_i(0)} E_{RN}\{g(\mathbf{S}(0),T)\} = \exp\{-rT\}\int g(\mathbf{Z}(t_1),\ldots,\mathbf{Z}(T))\frac{\partial}{\partial S_i(0)}\phi(\mathbf{S}(0),\mathbf{Z}(t_1),\ldots,\mathbf{Z}(T))d\mathbf{Z}(t_1)\ldots d\mathbf{Z}(T)$$

$$\frac{\partial}{\partial S_i(0)}\hat{g}(\mathbf{S}(0),T) = \exp\{-rT\}\sum_{n=1}^{N} g^n(.)\frac{1}{\phi}\frac{\partial\phi}{\partial S_i(0)}$$

$$\frac{1}{\phi}\frac{\partial\phi}{\partial S_i(0)} = \frac{\partial}{\partial S_i(0)}(\log\phi) \text{- score function}$$

Fast Computation of Gamma – Likelihood Ratio

- Take payoff *g* to be the pathwise delta (vector payoff) and evaluate the score function with respect to **S**(0). Likelihood ratio estimator is:

$$\Gamma_{ij}^{n} = \frac{\partial C^{n}}{\partial S_{j}(0)} \frac{\left(\mathbf{Z}(t_1)^{T}\mathbf{C}^{-1/2}\right)_{i}}{t_1 S_{i}(0)} - \left(\frac{1}{S_{i}(0)}\frac{\partial C^{n}}{\partial S_{i}(0)}\right) I_{[i=j]}$$

$$\hat{\Gamma}_{ij} = \exp(-rT)\frac{1}{N}\sum_{n=1}^{N}\Gamma_{ij}^{n}$$

- Score function only depends on density of path at at t_1 – the first ratchet point (Markov property).
- Will allow calculation of full Gamma matrix in same pass as liability.
- Variance in estimate will increase close to a ratchet point (small t_1) or small S(0).
- Not always more accurate than bump and reprice.

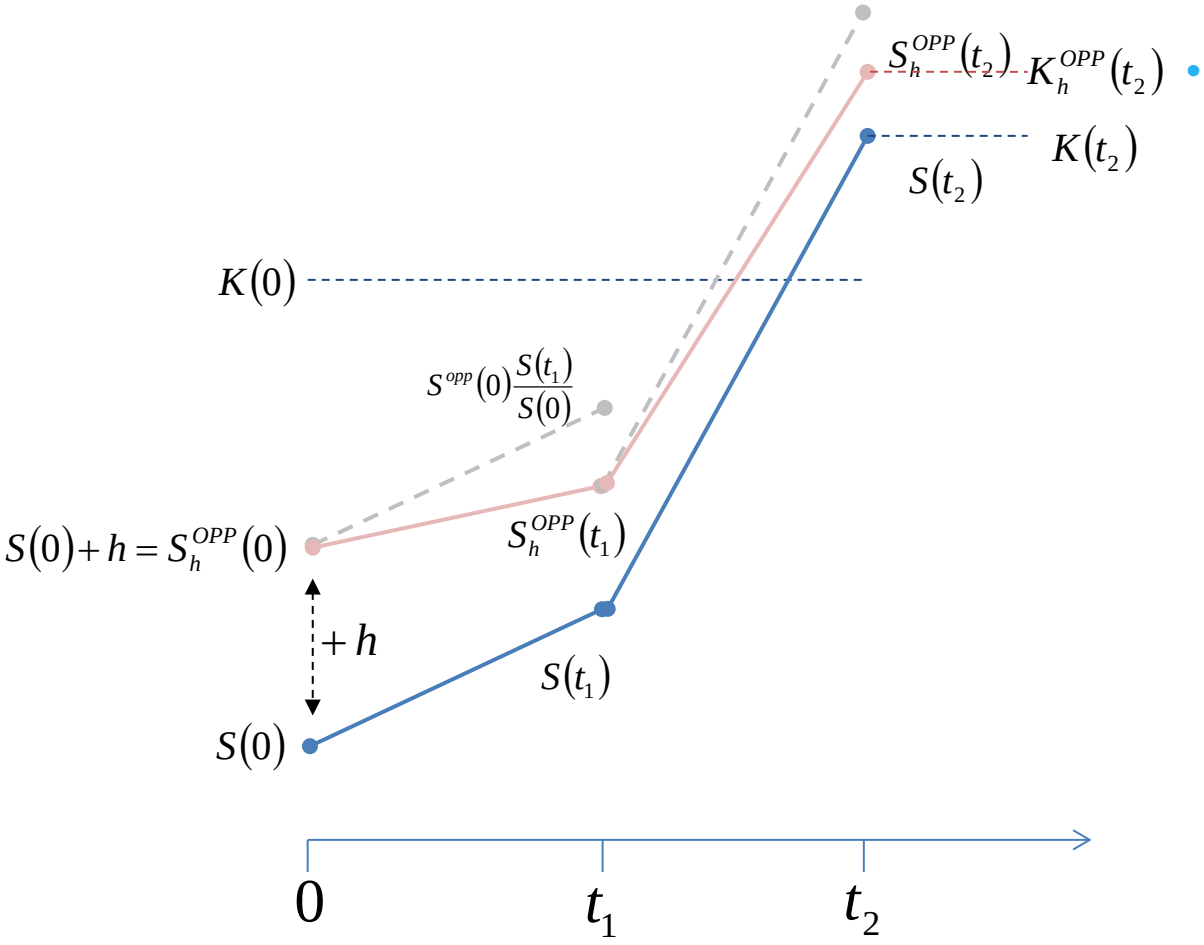
OPP Gamma Estimation

- Optimal Partial Proxy - Chan and Joshi 2014.
- Applicable to computing sensitivities of integrals where integrand is discontinuous.
- Constructs a perturbed asset path that eliminates discontinuities in the payoff.
- Optimal in sense that it minimises the variance of monte carlo weights introduced by perturbing the path.
- With discontinuities eliminated – small h limit can be taken just as in pathwise method.

OPP Gamma Estimation

- Applied to pathwise delta (as with likelihood ratio method).
- Perturb parameter Si(0) by h, then construct OPP path SOPP(t) from original path S(t) so that discontinuities in the payoff (the delta) are eliminated.
- Points of discontinuity are at ratchet points and expiry.
- Discontinuities will be eliminated if the perturbed path is on the same side of the strike as the original path at ALL ratchet points and at expiry. N.B not enough to be on same side of strike only at expiry, as the payoff is path dependent.

OPP Gamma Estimation



- Perturbation at each ratchet point, incurs monte carlo weight

$$w_h^{OPP}(t_n)$$

Gamma OPP Estimation

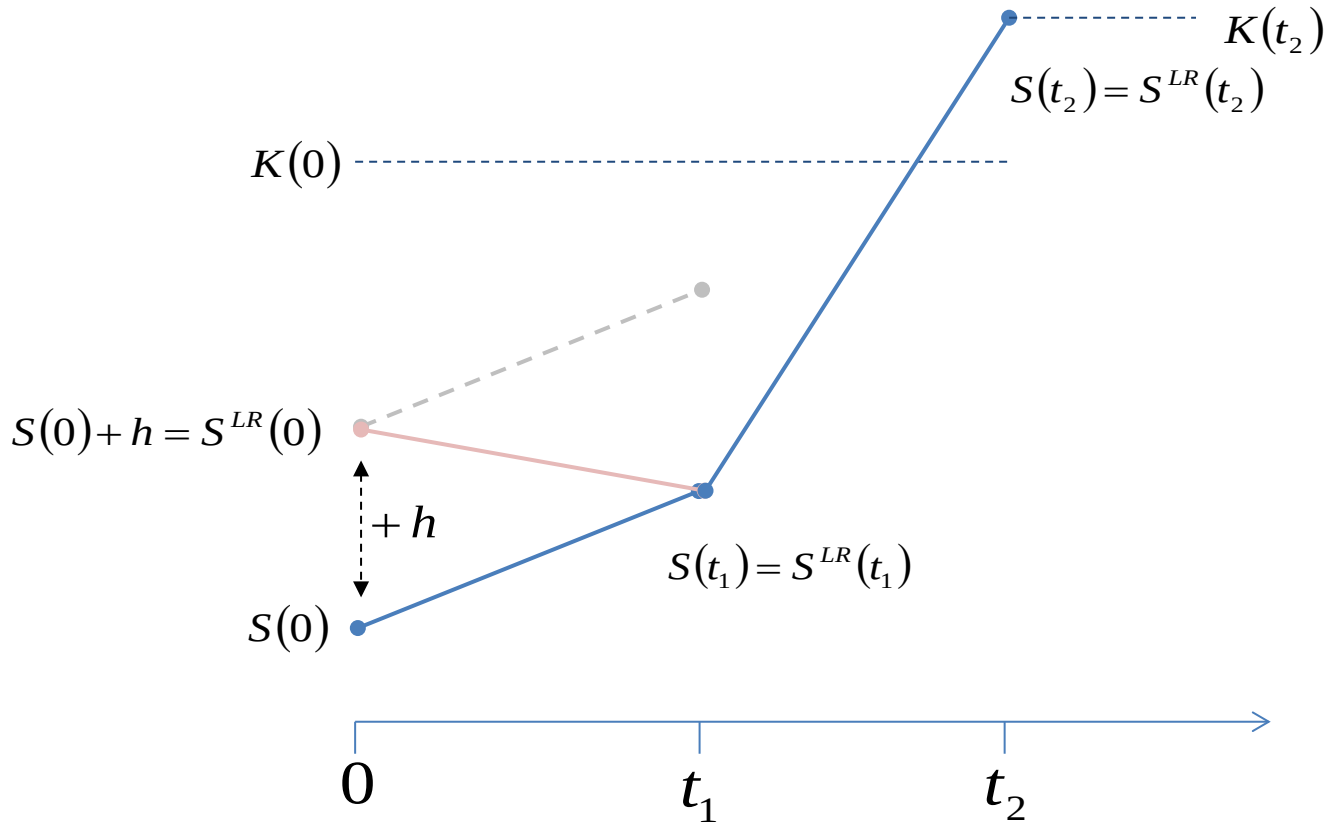
$$\frac{\partial}{\partial S_i(0)} E\{g(\mathbf{S}(T))\} = \lim_{h\to 0} E\left\{\frac{\prod_n w_h^{OPP}(t_n) g\left(\mathbf{S}_h^{OPP}(T)\right) - g(\mathbf{S}(T))}{h}\right\}$$

$$= E\left\{g(\mathbf{S}(T)) \frac{\partial}{\partial S_i(0)} \prod_n w^{OPP}(t_n) + \frac{\partial}{\partial S_i(0)} g\left(\mathbf{S}^{OPP}(T)\right)\right\}$$

- Application to delta to compute gamma not trivial.
- Computational savings great as well as significant increase in accuracy.
- Approx. 5 times number of operations to compute liability to also compute all delta components and full gamma matrix. Savings dependent on dimension of problem (from d>2, certainly more efficient)

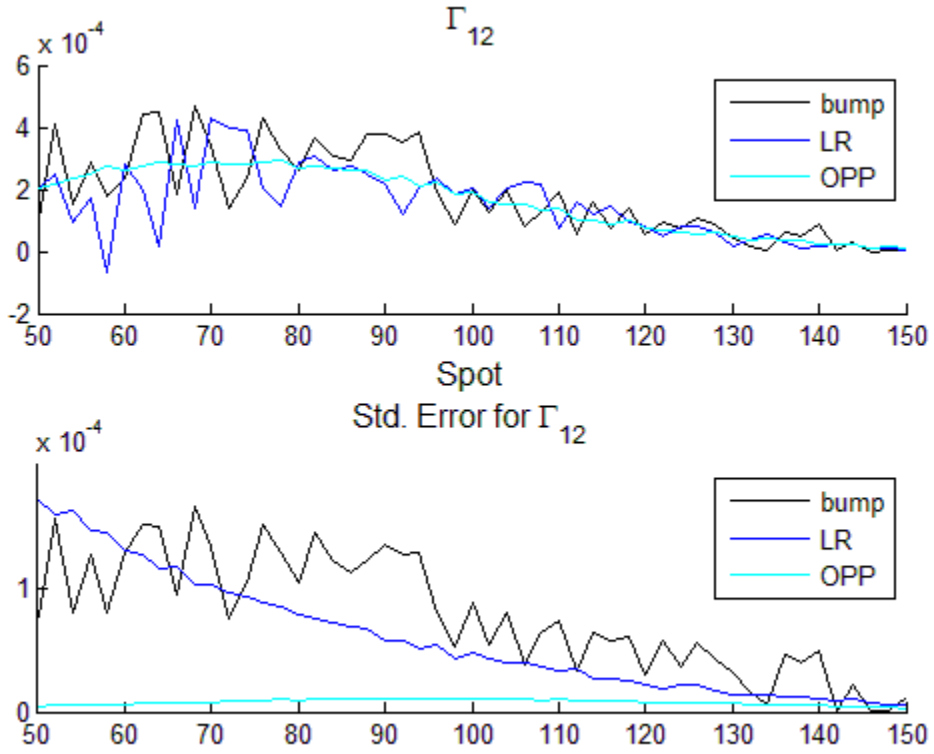
OPP Gamma Estimation

- Link to Likelihood ratio.
- As a re-sampling scheme, LR forces perturbed and unperturbed paths to be equal at first ratchet point. Not optimal in terms of variance of estimator.
- Derivative of monte-carlo weight is the score function



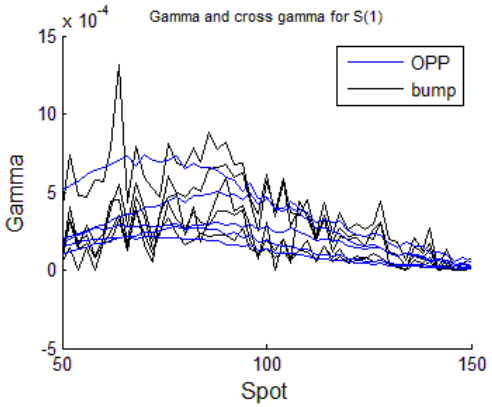
Simulation Examples

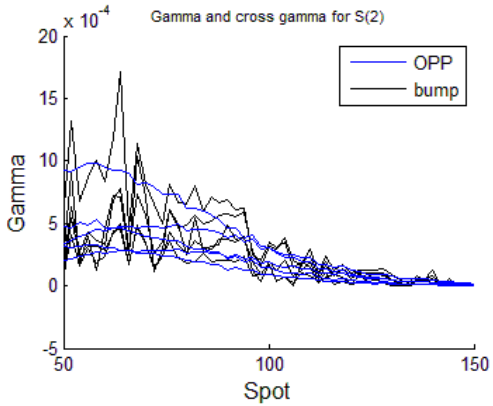
- Example with 5 assets in basket
- N = 1000 paths used, Gamma computed for varying initial Spot for each underlying.
- OPP method dominates both bump and re-price and Likelihood Ratio in terms of accuracy.

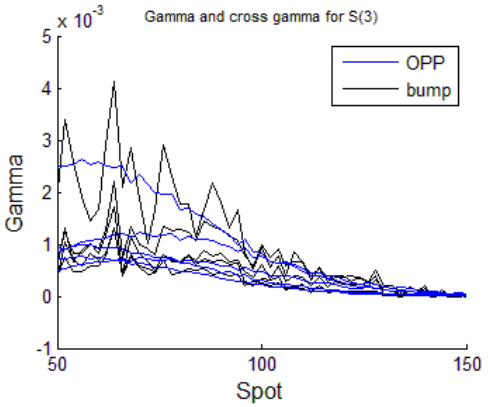


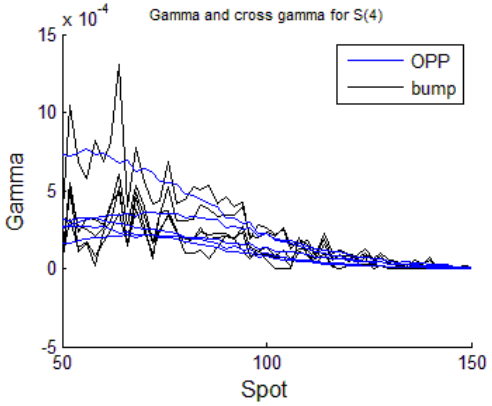
Simulation Examples

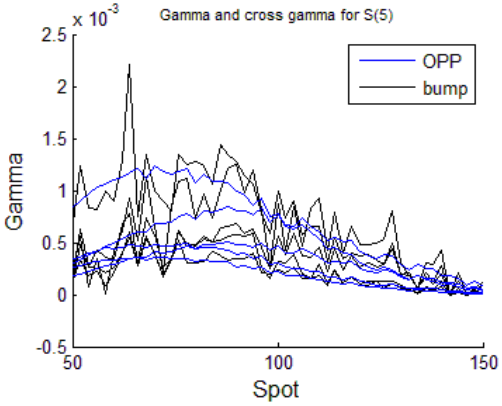
- OPP and Likelihood Ratio can compute Delta and full Gamma matrix using the information created when computing the liability. (more or less)



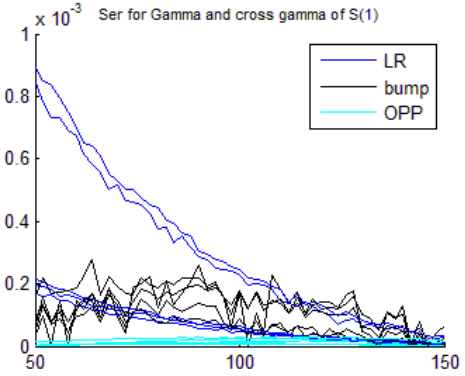


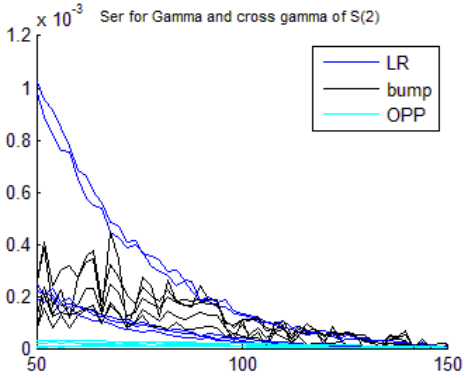


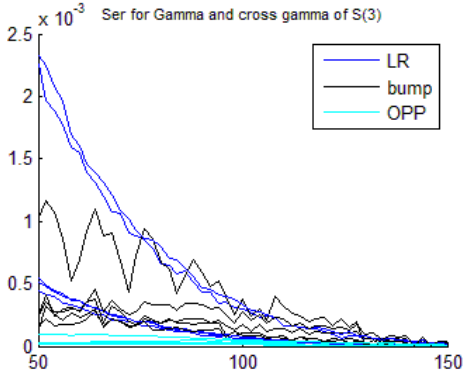


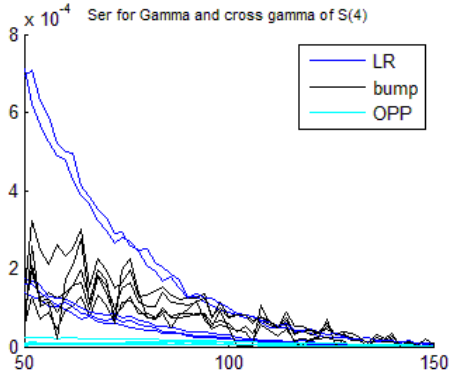


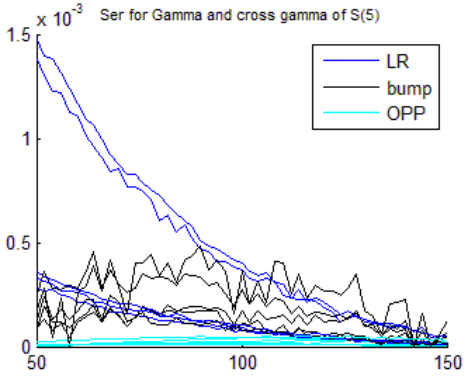
Simulation Examples





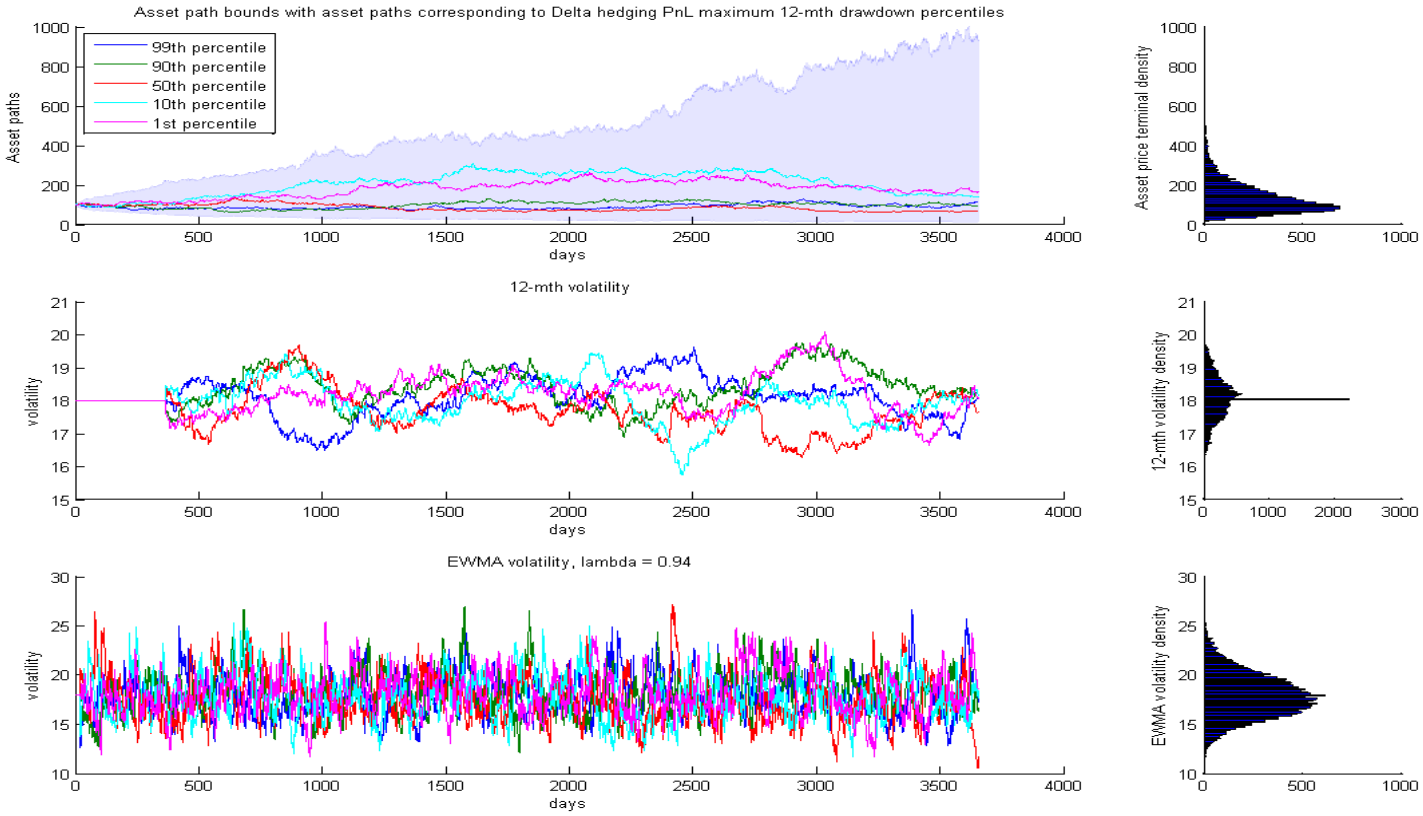






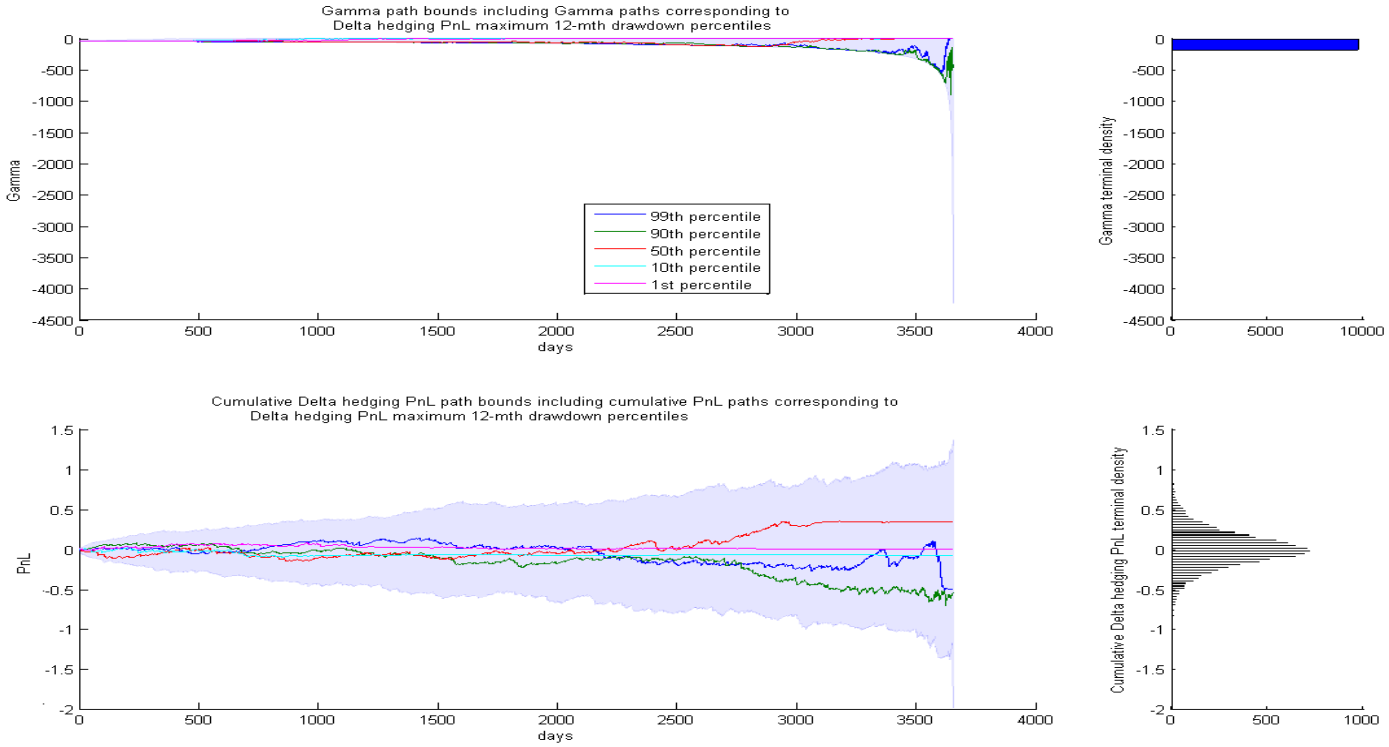
Forward Looking Risk Measure

- In model i.e. inner & outer loop GBM same diffusion parameter (18% p.a.) and no risk premium in the outer loop.



Forward Looking Risk Measure

- Expected losses across all possible future paths is zero because on average the outer loop paths have same diffusion parameter as inner loop.
- 99.5% ile 12 month drawdown is a scenario where in final year:
- realised volatility spikes to 25% for a short period near maturity
- the guarantee is very close to being at the money and is at peak gamma levels at the same time.



Conclusion

- Gaining deep insight into the future risk inherent in GMxB products requires us to quantify the DISTRIBUTION of the hedging P&L (liability minus hedge assets) at all points the future.

- To do this we need:
  - stochastic on stochastic simulations that project liability characteristics into the future with a high degree of accuracy
  - A platform for simulating a hedging strategy applied to liabilities projected into the future

OPP Gamma methodology presented today is a key enabler for developing a forward looking risk measure for GMxB's using stochastic on stochastic simulations.