Some advances on the Erlang(n) dual risk model

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Abstract

We consider the dual risk model, where premiums are regarded as costs and claims are viewed as profits. The surplus can be interpreted as a venture capital like the capital of an economic activity involved in research and development.

Main existing results on the subject assume that the waiting times between the profits arrivals follow an exponential distribution. That is like the classical Cramèr-Lunderg risk model. We make a generalization to the case when such times are Erlang(n) distributed, and we perform the calculations to obtain expressions for the ruin probability and the Laplace transform of the time of ruin for a general profit amount distribution, using the roots of the fundamental and the generalized Lundberg's equation. For the expected discounted dividends we do the same developments for the case when the common profits follow a Phase–Type PH(m) distribution.

Finally, for some particular profit distributions we show examples and obtain numerical results.

Keywords: dual risk model; Erlang(n) interclaim times; generalized Lundberg's equation; ruin probability; time of ruin; expected discounted dividends.

1 Introduction

We consider the dual risk model where the surplus or equity of the company is commonly described as

$$U(t) = u - ct + \sum_{i=1}^{N(t)} X_i,\ t \geq 0,\ u \geq 0,$$

where u is the initial surplus and c is a constant meaning the rate of expenses, assumed deterministic and fixed. The gains are represented by the sequence $\{X_i\}$ of i.i.d. random variables with cumulative distribution function $P(x)$ and density $p(x)$. Denote by $\mu_k = E[X_1^k]$ the k-th moment of X_i. Existence of μ_1 is a basic assumption fin this model. Existence of Higher moments may be needed, this will become apparent in the manuscript. Let $\hat{p}(s)$ be the Laplace transform of $p(.)$.

By $N(t) = \max\{k :\ T_1 + T_2 + \cdots + T_k \leq t\}$ we denote the number of gains up to time t, where the random variable T_i denote the interclaim time between the $i-1$-th and the i-th claim. $\{T_i\}$ is a sequence of i.i.d. random variables and is independent of $\{X_i\}$. We work here in the case were those interclaim times are Erlang(n, λ) distributed, therefore with density

$$k_n(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, \quad t \geq 0,\ \lambda > 0,\ n \in \mathbb{N}^+.$$

In the classical primal model, the usual Sparre–Andersen risk model with application in insurance, a fundamental condition is the positive loading, that is $cE(W_1) > E(X_1)$. On the contrary, on the dual risk model we discuss here, we have that condition reversed, we might call it a negative loading, i.e., $cE(W_1) < E(X_1)$ giving the inequality $cn < \lambda\mu_1$. So on average, our gains should be superior to the expenses, otherwise ruin will be certain.

This model has been having increasing interest in risk theory in recent times. There are many possible interpretations for this dual model. We can treat the surplus as the amount of capital of a business engaged in research and development, where gains are random and at random instants, and costs are certain. More precisely, the company pays expenses which occurs continuously along time for the research activity and gets on occasionas profits according to an Erlang(n) distribution.

We can now consider some of the basic definitions and notations for the dual risk model, those which we address through this paper. Let

$$\tau_u = \begin{cases} inf\{t > 0 : U(t) = 0 \mid U(0) = u\} \\ \infty, \quad \text{if } U(t) \geq 0 \ \forall t \geq 0 \end{cases}$$

be the time to ruin, let $\psi(u) = P(\tau_u < \infty)$ be the ultimate ruin probability and

$$\psi(u,\delta) = E[e^{-\delta\tau_u}\mathbb{I}(\tau_u < \infty) \mid U(0) = u]$$

be the Laplace transform of the time to ruin, where $\delta > 0$ is the force of interest.

Now let b denote a fixed barrier level, $b \geq u$. Let $\{D_i\}_{i=1}^{\infty}$ be the sequence of dividend payments, where payments are made each time the surplus upcrosses the barrier b, and D_u be the aggregate discounted dividends, from initial surplus u. We denote by $V(u,b) = E[D_u]$ the expected present value at force of interest δ of dividends payable to shareholders prior to ruin and by $V_m(u,b) = E[D_u^m]$, $m \geq 1$, the higher order moments of D_u. For simplicity on the notation we denote $V_1(u,b) = V(u,b)$. For details on the dividends process and definitions please see Afonso et al (2011).

2 The Lundberg's equations

For the Erlang(n) dual risk model, the fundamental Lundberg's equation comes in the form as

$$\left(1-\left(\frac{c}{\lambda}\right)s\right)^n=\hat{p}(s) \tag{2.1}$$

and the generalized Lundberg's equation, for the force of interest $\delta>0$ becomes

$$\left(1+\frac{\delta}{\lambda}-\left(\frac{c}{\lambda}\right)s\right)^n=\hat{p}(s) \tag{2.2}$$

According to Li and Garrido (2004), in the primal Sparre–Andersen risk model with Erlang(n) distributed interclaim times the equation (2.2) has n roots with positive real parts and equation (2.1) has $n-1$ roots with positive real parts. However, In the dual model this isn't exactly the same, since both equations will have n roots with positive real parts. Let $\rho_1(\delta),\ldots,\rho_n(\delta)$ denote these roots for $\delta>0$. To see this, we just proceed like in Li and Garrido (2004). Define the function $h(s)=\left(\frac{\lambda}{c}\right)^n\hat{p}(s)-\left(\frac{\lambda+\delta}{c}-s\right)^n$. Since $h(0)<0$ and $\lim_{s\to\infty}h(s)=+\infty$, then for a sufficiently smooth density $p(x)$ we will have at least one negative root, denoted $-R(\delta)$, for $R(\delta)>0$. Now,

$$h'(0)=-\left(\frac{\lambda}{c}\right)^n\mu_1+n\left(\frac{\lambda+\delta}{c}\right)^{n-1}=\left(\frac{\lambda}{c}\right)^{n-1}\left(-\frac{\lambda}{c}\mu_1\right)+n\left(\frac{\lambda+\delta}{c}\right)^{n-1}<0\,,$$

for the negative loading condition ($cn<\lambda\mu_1$) and for a sufficiently small δ. This means that $h(s)$ must have a local maximum between $-R(\delta)$ and 0, so $\lim_{\delta\to0^+}(-R(\delta))=0$, and all of the $\rho_1(\delta),\ldots,\rho_n(\delta)$ remain different than zero in the case $\delta=0$ as well.

Remark 2.1 *The roots $\rho_1(\delta),\ldots,\rho_n(\delta)$ are all distinct for $\delta\geq0$, see Ji and Zhang (2011).*

3 Ruin probabilities

The ruin probability in the dual risk model with exponential interclaim times $(k(t) = \lambda e^{-\lambda t})$, satisfies the following renewal equation

$$\psi(u) = e^{-\lambda t_0} + \int_0^{t_0} \lambda e^{-\lambda t} \int_0^{\infty} p(x)\psi(u - ct + x)dx\, dt \tag{3.1}$$

where $t_0 = u/c$ is the time to reach the ruin level without any single claim. This can be found in Afonso et al (2011). Differentiation with respect to u gives an integro–differential equation for $\psi(u)$

$$\psi(u) + \left(\frac{c}{\lambda}\right)\frac{d}{du}\psi(u) = \int_0^{\infty} p(x)\psi(u+x)dx$$

We can write this equation as

$$\left(I + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)\psi(u) = \int_0^{\infty} p(x)\psi(u+x)dx \tag{3.2}$$

where I is the identity operator and $\mathcal{D}$ is the differentiation operator (with respect to u).

For the dual risk model with Erlang(n) interclaim times the renewal equation corresponding to (3.1) becomes

$$\psi(u) = 1 - K_n(t_0) + \int_0^{t_0} k_n(t) \int_0^{\infty} p(x)\psi(u - ct + x)dx\, dt \tag{3.3}$$

and we give the integro–differential equation analogous to (3.2) in the following theorem

Theorem 3.1 *In the Erlang(n) dual risk model the ruin probability satisfies the integro–differential equation*

$$\left(I + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n \psi(u) = \int_0^{\infty} p(x)\psi(u+x)dx \tag{3.4}$$

with boundary conditions

$$\psi(0) = 1 \quad \textit{and} \quad \left.\frac{d^i}{du^i}\psi(u)\right|_{u=0} = 0,\ i = 1,\ldots,n-1. \tag{3.5}$$

Proof. We proceed taking successive derivatives of the ruin probability using the renewal equation (3.3). Changing variables the renewal equation can be rewritten in the form

$$\psi(u) = 1 - K_n\left(\frac{u}{c}\right) + \frac{1}{c}\int_0^u k_n\left(\frac{u-s}{c}\right)W(s)ds,$$

where $W(s) = \int_0^\infty \psi(s+x)p(x)dx$. After applying the operator $\left(I + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)$ to the ruin probability we get

$$\left(I + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)\psi(u) = 1 - K_{n-1}\left(\frac{u}{c}\right) + \frac{1}{c}\int_0^u k_{n-1}\left(\frac{u-s}{c}\right)W(s)ds.$$

Following an inductive argument, it is easy to show that

$$\left(I + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^i\psi(u) = 1 - K_{n-i}\left(\frac{u}{c}\right) + \frac{1}{c}\int_0^u k_{n-i}\left(\frac{u-s}{c}\right)W(s)ds,$$

for $i = 1,\ldots,n-1$. In particular, for $i = n-1$ we obtain

$$\left(I + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^{n-1}\psi(u) = 1 - K_1\left(\frac{u}{c}\right) + \frac{1}{c}\int_0^u k_1\left(\frac{u-s}{c}\right)W(s)ds.$$

Applying once more the operator gives

$$\left(I + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n\psi(u) = W(u).$$

This proves equation (3.4). We have been using here some very known properties of the

Erlang(n) probability density function, namely

$$\begin{aligned} k_n'(t) &= \lambda(k_{n-1}(t) - k_n(t)), \\ k_n^{(i)}(0) &= 0, \quad i = 0, \ldots, n-2, \\ k_n^{(n-1)}(0) &= \lambda^n. \end{aligned}$$

For the boundary conditions, clearly $\psi(0) = 1$ and we find the remaining conditions computing directly the derivatives of $\psi(u)$ and evaluating at $u = 0$.

$$\frac{d^i}{du^i}\psi(u) = -\left(\frac{1}{c}\right)^i k_n^{(i-1)}\left(\frac{u}{c}\right) + \left(\frac{1}{c}\right)^{i+1} \int_0^u k_n^{(i)}\left(\frac{u-s}{c}\right) W(s)ds + \left(\frac{1}{c}\right)^i k_n^{(i-1)}(0)W(u)$$

for $i = 1, \ldots, n-1$, so we obtain $\left.\frac{d^i}{du^i}\psi(u)\right|_{u=0} = 0,\ i = 1, \ldots, n-1.$ ■

Theorem 3.2 *The ruin probability can be written as a combination of exponential functions*

$$\psi(u) = \sum_{k=1}^{n} (-1)^{k-1} \left[\frac{\prod_{i=1, i\neq k}^{n} \rho_i}{\left(\prod_{i=1}^{k-1} (\rho_k - \rho_i)\right)\left(\prod_{j=k+1}^{n} (\rho_j - \rho_k)\right)} \right] e^{-\rho_k u}, \tag{3.6}$$

where $\rho_1, \ldots, \rho_n$ *are the only roots of the fundamental Lundberg's equation* (2.1) *which have positive real parts.*

Proof. To see this, we first look for particular solutions of the integro–differential equation (3.4). Let

$$\left(I + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n f(u) = \int_0^\infty p(x) f(u+x)dx, \tag{3.7}$$

be the equation for a general solution $f(u)$.

Let $f(u) = e^{-ru}$, for some $r \in \mathbb{C}$. Then on the left hand side of (3.4) we obtain

$$\left(I + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n f(u) = \left(1 - \left(\frac{c}{\lambda}\right)r\right)^n e^{-ru}$$

while on the right hand side

$$\int_0^\infty p(x)f(u+x)dx = \int_0^\infty p(x)e^{-r(u+x)}dx = e^{-ru}\hat{p}(r)\ .$$

In the equality we get

$$\left(1 - \left(\frac{c}{\lambda}\right)r\right)^n = \hat{p}(r)\,,$$

which means that r must be a root of the Fundamental Lundberg's equation (2.1).

Let $\rho_1, \ldots, \rho_n$ be the n solutions of the (2.1) that have positive real parts. Define the functions $f_1(u) = e^{-\rho_1 u}, \ldots, f_n(u) = e^{-\rho_n u}$. Since they are linearly independent we can write any solution of (3.7) in the form

$$f(u) = \sum_{i=1}^n a_i e^{-\rho_i u}, \quad a_i \quad \text{constants.}$$

Using the boundary conditions (3.5) the constants a_i can be determined by solving a system of n equations on the unknowns a_i, which in matrix form looks like this

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \rho_1 & \rho_2 & \cdots & \rho_n \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1^{n-1} & \rho_2^{n-1} & \cdots & \rho_n^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

or $\overrightarrow{a}^t = P^{-1}\overrightarrow{e}^t$, where $P = P(\rho_1, \ldots, \rho_n)$ is a Vandermonde matrix , $\overrightarrow{a} = \{a_1, a_2, \ldots, a_n\}$ and $\overrightarrow{e} = \{1, 0, \ldots, 0\}$.

The determinant of P is

$$DetP = \prod_{1\leq i<j\leq n} (\rho_j - \rho_i)$$

and we get expressions for the coefficients

$$\begin{aligned} a_k &= \frac{(-1)^{k-1}(\prod_{i=1,\, i\neq k}^{n} \rho_i)(\prod_{1\leq i<j\leq n,\, i\neq j,\, j\neq k}(\rho_j - \rho_i))}{\prod_{1\leq i<j\leq n}(\rho_j - \rho_i)} \\ &= \frac{(-1)^{k-1}(\prod_{i=1,\, i\neq k}^{n} \rho_i)}{(\prod_{i=1}^{k-1}(\rho_k - \rho_i))(\prod_{j=k+1}^{n}(\rho_j - \rho_k))} \end{aligned}$$

Hence we get the result. ∎

Example 3.1 *For $n = 1$:*

In the exponential case, Gerber (1979) found that $\psi(u) = e^{-\rho u}$, where ρ is the unique positive root of the fundamental Lundberg's equation (2.1).

For $n = 2$:

$$\psi(u) = \frac{\rho_2}{\rho_2 - \rho_1} e^{-\rho_1 u} - \frac{\rho_1}{\rho_2 - \rho_1} e^{-\rho_2 u},$$

where $\rho_1, \rho_2 > 0$ are real and solutions of $\left(1 - \left(\frac{c}{\lambda}\right) s\right)^2 = \hat{p}(s)$.

For $n = 3$:

$$\psi(u) = \frac{\rho_2\rho_3}{(\rho_3 - \rho_1)(\rho_2 - \rho_1)} e^{-\rho_1 u} - \frac{\rho_1\rho_3}{(\rho_3 - \rho_2)(\rho_2 - \rho_1)} e^{-\rho_2 u} + \frac{\rho_1\rho_2}{(\rho_3 - \rho_1)(\rho_3 - \rho_2)} e^{-\rho_3 u},$$

where ρ_1, ρ_2, ρ_3 are solutions of $\left(1 - \left(\frac{c}{\lambda}\right) s\right)^3 = \hat{p}(s)$, one root is real and the other 2 are complex conjugates.

4 Laplace transforms

For the Erlang(n) case, the Laplace transform of the time of ruin

$$\psi(u,\delta) = E[e^{-\delta\tau_u}\mathbb{I}(\tau_u < \infty) \mid U(0) = u], \quad \delta > 0,$$

satisfies the renewal equation

$$\psi(u,\delta) = \left(1 - K_n\left(\frac{u}{c}\right)\right) e^{-\delta\left(\frac{u}{c}\right)} + \frac{1}{c}\int_0^u k_n\left(\frac{u-s}{c}\right) e^{-\delta\left(\frac{u-s}{c}\right)} \int_0^\infty p(x)\psi(s+x,\delta)dx\,dt. \tag{4.1}$$

Theorem 4.1 *In the Erlang(n) dual risk model the Laplace transform of the time of ruin satisfies the integro–differential equation*

$$\left(\left(1+\frac{\delta}{\lambda}\right) I + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n \psi(u,\delta) = \int_0^\infty p(x)\psi(u+x,\delta)dx, \tag{4.2}$$

with boundary conditions

$$\psi(0,\delta) = 1, \quad \frac{d^i}{du^i}\psi(u,\delta)\Big|_{u=0} = (-1)^i \left(\frac{\delta}{c}\right)^i, \; i = 1,\ldots,n-1. \tag{4.3}$$

Proof. We proceed, like in the ruin probability case, taking successive derivatives of the Laplace transform using the renewal equation (4.1). Changing variables the renewal equation can be rewritten in the form

$$\psi(u,\delta) = \left(1 - K_n\left(\frac{u}{c}\right)\right) e^{-\delta\left(\frac{u}{c}\right)} + \frac{1}{c}\int_0^u k_n\left(\frac{u-s}{c}\right) e^{-\delta\left(\frac{u-s}{c}\right)} W(s)ds,$$

where $W(s) = \int_0^\infty \psi(s+x,\delta)p(x)dx$.

After applying the operator $\left(\left(1+\frac{\delta}{\lambda}\right) I + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)$ to the Laplace transform we get

$$\left(1-K_{n-1}\left(\frac{u}{c}\right)\right)e^{-\delta(\frac{u}{c})}+\frac{1}{c}\int_0^u k_{n-1}\left(\frac{u-s}{c}\right)e^{-\delta(\frac{u-s}{c})}W(s)ds.$$

Analogously, following an inductive argument, we show that

$$\left(\left(1+\frac{\delta}{\lambda}\right)I+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^i\psi(u)=\left(1-K_{n-i}\left(\frac{u}{c}\right)\right)e^{-\delta(\frac{u}{c})}+\frac{1}{c}\int_0^u k_{n-i}\left(\frac{u-s}{c}\right)e^{-\delta(\frac{u-s}{c})}W(s)ds,$$

for $i=1,\ldots,n-1$. In particular, for $i=n-1$ we obtain

$$\left(\left(1+\frac{\delta}{\lambda}\right)I+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^{n-1}\psi(u)=\left(1-K_1\left(\frac{u}{c}\right)\right)e^{-\delta(\frac{u}{c})}+\frac{1}{c}\int_0^u k_1\left(\frac{u-s}{c}\right)e^{-\delta(\frac{u-s}{c})}W(s)ds.$$

Applying once more the operator gives

$$\left(\left(1+\frac{\delta}{\lambda}\right)I+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n\psi(u)=W(u).$$

This proves equation (4.2).

For the boundary conditions, clearly $\psi(0,\delta)=1$ and we find the remaining conditions computing directly the derivatives of $\psi(u,\delta)$ and evaluating at $u=0$.

$$\begin{aligned}\frac{d^i}{du^i}\psi(u,\delta) \;=\; & \left[\left(-\frac{\delta}{c}\right)^i\left(1-K_n\left(\frac{u}{c}\right)\right)+\cdots-\left(\frac{1}{c}\right)^i k_n^{(i-1)}\left(\frac{u}{c}\right)\right]e^{-\delta(\frac{u}{c})}+\\ & \left(\frac{1}{c}\right)\int_0^u\left[\left(-\frac{\delta}{c}\right)^i k_n\left(\frac{u-s}{c}\right)+\cdots+\left(\frac{1}{c}\right)^i k_n^{(i)}\left(\frac{u-s}{c}\right)\right]e^{-\delta(\frac{u-s}{c})}W(s)ds+\\ & \left(\frac{1}{c}\right)\left[\left(-\frac{\delta}{c}\right)^{i-1}k_n(0)+\cdots+\left(\frac{1}{c}\right)^{i-1}k_n^{(i-1)}(0)\right]W(u)\end{aligned}$$

for $i=1,\ldots,n-1$, so we obtain $\left.\frac{d^i}{du^i}\psi(u,\delta)\right|_{u=0}=\left(-\frac{\delta}{c}\right)^i$, $i=1,\ldots,n-1$. ■

Theorem 4.2 *The Laplace transform of the time of ruin can be written as a combination of* *exponential functions*

$$\psi(u,\delta)=\sum_{k=1}^{n}(-1)^{k-1}\left[\frac{\prod_{i=1,i\neq k}^{n}\left(\rho_i-\frac{\delta}{c}\right)}{\left(\prod_{i=1}^{k-1}(\rho_k-\rho_i)\right)\left(\prod_{j=k+1}^{n}(\rho_j-\rho_k)\right)}\right]e^{-\rho_k u}, \tag{4.4}$$

where $\rho_1,\ldots,\rho_n$ *are the only roots of the Lundberg's equation* (2.2) *which have positive real parts.*

Proof. With a similar procedure like in the ruin probability, we obtain formula (4.4). All the functions $e^{-\rho_k u}$'s, for $\rho_k \in \{\rho_1,\ldots,\rho_n\}$, are solutions of the integro–differential equation

$$\left(\left(1+\frac{\delta}{\lambda}\right)I+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n f(u)=\int_0^\infty p(x)f(u+x)dx, \tag{4.5}$$

Since these functions are linearly independent, we can write every solution of (4.5) as a linear combination of them. Therefore

$$\psi(u,\delta)=\sum_{i=1}^{n}b_i e^{-\rho_i u}, \quad b_i \quad \text{constants.}$$

and we obtain the constants solving the following matrix system

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}=\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \rho_1 & \rho_2 & \cdots & \rho_n \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1^{n-1} & \rho_2^{n-1} & \cdots & \rho_n^{n-1} \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ \frac{\delta}{c} \\ \vdots \\ \left(\frac{\delta}{c}\right)^{n-1} \end{pmatrix}$$

or $\overrightarrow{b}^t=P^{-1}\overrightarrow{C}^t$, where $P=P(\rho_1,\ldots,\rho_n)$ is a Vandermonde matrix , $\overrightarrow{b}=\{b_1,b_2,\ldots,b_n\}$ and $\overrightarrow{C}=\left\{1,\frac{\delta}{c},\ldots,\left(\frac{\delta}{c}\right)^{n-1}\right\}$.

Finally we get expressions for the coefficients

$$
\begin{aligned}
b_k &= \frac{(-1)^{k-1}(\prod_{i=1,\, i\neq k}^{n}(\rho_i-\frac{\delta}{c}))(\prod_{1\leq i<j\leq n,\, i\neq j,\, j\neq k}(\rho_j-\rho_i))}{\prod_{1\leq i<j\leq n}(\rho_j-\rho_i)} \\
&= \frac{(-1)^{k-1}(\prod_{i=1,\, i\neq k}^{n}(\rho_i-\frac{\delta}{c}))}{(\prod_{i=1}^{k-1}(\rho_k-\rho_i))(\prod_{j=k+1}^{n}(\rho_j-\rho_k))}
\end{aligned}
$$

Hence we get the result. ■

Example 4.1 *For $n=1$:*

In the exponential case, Ng (2009) found that $\psi(u,\delta)=e^{-R_\delta u}$, where R_δ is the unique positive root of the generalized Lundberg's equation (2.2).

For $n=2$:

$$\psi(u)=\frac{\rho_2-\frac{\delta}{c}}{\rho_2-\rho_1}e^{-\rho_1 u}-\frac{\rho_1-\frac{\delta}{c}}{\rho_2-\rho_1}e^{-\rho_2 u},$$

where $\rho_1,\rho_2>0$ *are real and solutions of* $\left(1+\frac{\delta}{\lambda}-\left(\frac{c}{\lambda}\right)s\right)^2=\hat{p}(s).$

For $n=3$:

$$\psi(u)=\frac{(\rho_2-\frac{\delta}{c})(\rho_3-\frac{\delta}{c})}{(\rho_3-\rho_1)(\rho_2-\rho_1)}e^{-\rho_1 u}-\frac{(\rho_1-\frac{\delta}{c})(\rho_3-\frac{\delta}{c})}{(\rho_3-\rho_2)(\rho_2-\rho_1)}e^{-\rho_2 u}+\frac{(\rho_1-\frac{\delta}{c})(\rho_2-\frac{\delta}{c})}{(\rho_3-\rho_1)(\rho_3-\rho_2)}e^{-\rho_3 u},$$

where ρ_1,ρ_2,ρ_3 *are solutions of* $\left(1+\frac{\delta}{\lambda}-\left(\frac{c}{\lambda}\right)s\right)^3=\hat{p}(s),$ *one root is real and the other 2 are complex conjugates.*

5 Expected Discounted Dividends

For the exponential case, the expected present value of the discounted dividends, $V(u,b)$, satisfies the renewal equation

$$\begin{aligned} V(u,b) &= \int_0^{\frac{u}{c}} \lambda e^{-(\lambda+\delta)t} \left[\int_0^{b-u+ct} V(u-ct+y,b)p(y)dy + \right. \\ &\quad \left. \int_{b-u+ct}^{\infty} (y+u-ct-b+V(b,b))p(y)dy \right] dt \end{aligned}$$

for $u \leq b$.

Note that $V(0,b) = 0$, since at $u = 0$ we have ruin, and that

$$V(u,b) = u - b + V(b,b), \quad \text{for} \quad u > b. \tag{5.1}$$

Differentiating with respect to u gives

$$\left(\left(1 + \frac{\delta}{\lambda}\right) I + \left(\frac{c}{\lambda}\right) \mathcal{D} \right) V(u,b) = W(u,b),$$

where

$$W(u,b) = \int_0^{b-u} V(u+y,b)p(y)dy + \int_{b-u}^{\infty} (y+u-b+V(b,b))p(y)dy.$$

In the Erlang(n) model ($n \geq 2$), the renewal equation is

$$\begin{aligned} V(u,b) &= \int_0^{\frac{u}{c}} k_n(t) e^{-(\delta)t} \left[\int_0^{b-u+ct} V(u-ct+y,b)p(y)dy + \right. \\ &\quad \left. \int_{b-u+ct}^{\infty} (y+u-ct-b+V(b,b))p(y)dy \right] dt. \end{aligned}$$

After changing variables we can write it in the following form

$$V(u,b) = \frac{1}{c}\int_0^u k_n\left(\frac{u-s}{c}\right)e^{-\delta(\frac{u-s}{c})}W(s,b)ds\ . \tag{5.2}$$

Theorem 5.1 $V(u,b)$ *satisfies the integro–differential equation*

$$\left(\left(1+\frac{\delta}{\lambda}\right)I+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n V(u,b) = W(u,b), \tag{5.3}$$

with boundary conditions

$$\left.\frac{d^i}{du^i}V(u,b)\right|_{u=0} = 0, \quad i = 0,\ldots,n-1. \tag{5.4}$$

Proof. The proof follows the same method applied before, taking successive derivatives of (5.2). ■

Because of condition (5.1), we can not write the solutions of (5.3) as a linear combination of n exponential functions as we did before in the cases of the ruin probability and the Laplace transform of the time of ruin. We will need instead, more than n exponentials, the exact number needed will depend on the nature of the distribution of the gains, $p(x)$.

5.1 The annihilator of $p(x-u)$

Note that we can rewrite $W(u,b)$ in the form

$$\begin{aligned} W(u,b) &= \int_u^b V(x,b)p(x-u)dx + \int_b^\infty (x-b+V(b,b))p(x-u)dx \\ & \int_u^b V(x,b)p(x-u)dx + \int_b^\infty \widetilde{V}(u,b)p(x-u)dx\ . \end{aligned} \tag{5.5}$$

The idea is to find a linear differential operator that will annihilate $p(x-u)$ (where the variable is u), so when we apply this operator to the integro–differential equation (5.3) we obtain a linear homogeneous differential equation of a higher degree (and the integral term vanish).

Lets consider the case when the gains follow a Phase-Type PH(m) distribution. Let

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \end{pmatrix}$$

be the matrix of transition rates between the transient states, let $\overrightarrow{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be the vector of the initial probabilities, $\overrightarrow{\eta} = (\eta_1, \eta_2, \ldots, \eta_m)$ the vector of the exit rates to the absorbing state, and $\overrightarrow{\gamma} = (1, 1, \ldots, 1)$ the $1 \times m$ vector with 1's in all the entries. Let I_m denote the identity matrix $m \times m$. It is well known that $p(x) = \overrightarrow{\alpha} e^{Bx} \overrightarrow{\eta}^t$ and that $P(x) = 1 - \overrightarrow{\alpha} e^{Bx} \overrightarrow{\gamma}^t$

Theorem 5.2 *One annihilator of degree m for $p(x-u)$ is $q_B(-D)$, where $D = \frac{d}{du}$ denote differentiation with respect to u and $q_B(y) = det(B - yI_m)$ is the characteristic polynomial of the matrix B.*

Proof. The proof is based on the Cayley–Hamilton theorem of linear algebra, which states that every square matrix satisfies its own characteristic equation. ■

Example 5.1 • *For the exponential distribution:*

We have $p(x) = \beta e^{-\beta x}$, so $B = (-\beta)$, $\overrightarrow{\alpha} = (1)$, $\overrightarrow{\eta} = (\beta)$ and $\overrightarrow{\gamma} = (1)$. Then

$$q_B(y) = det(B - yI_1) = -\beta - y \quad and \quad q_B(-D) = \frac{d}{du} - \beta.$$

It is easy to verify that $\left(\frac{d}{du} - \beta\right) p(x-u) = 0$.

- *For the Erlang(n) distribution:*

  We have $p(x) = \dfrac{\beta^n x^{n-1} e^{-\beta x}}{(n-1)!}$, *so*

$$B = \begin{pmatrix} -\beta & \beta & \cdots & 0 \\ 0 & -\beta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\beta \end{pmatrix},$$

  $\overrightarrow{\alpha} = (1,0,\ldots,0)$, $\overrightarrow{\eta} = (0,0,\ldots,0,\beta)$ *and* $\overrightarrow{\gamma} = (1,1,\ldots,1)$. *Then*

$$q_B(y) = det(B - yI_1) = (-\beta - y)^n \quad and \quad q_B(-D) = \left(\frac{d}{du} - \beta\right)^n.$$

  It is easy to verify that $\left(\frac{d}{du} - \beta\right)^n p(x-u) = 0$.

Now we want to apply $q_B(-D)$ to the integro–differential equation (5.3). Let

$$q_B(-D) = \sum_{i=0}^{m} q_i \frac{d^i}{du^i}, \quad q_i \quad \text{constants}$$

be the polynomial expression of $q_B(-D)$. Then

Theorem 5.3 *After applying* $q_B(-D)$ *to the integro–differential equation* (5.3) *we get a lin-*

ear homogeneous differential equation of degree $m+n$ of the following form

$$
\begin{aligned}
0 &= q_B(-D)\left[\left(\left(1+\frac{\delta}{\lambda}\right)I+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n V(u,b)-W(u,b)\right] \\
&= \sum_{l=0}^{n+m}\left[\sum_{i+k=l} q_i \binom{n}{n-k}\left(1+\frac{\delta}{\lambda}\right)^{n-k}\left(\frac{c}{\lambda}\right)^k\right]\frac{d^l}{du^l}V(u,b)+ \\
&+ \overrightarrow{\alpha}\,\overrightarrow{\eta}^t\sum_{j=0}^{m-1} q_{j+1}\frac{d^j}{du^j}V(u,b)\,. \qquad (5.6)
\end{aligned}
$$

5.2 The expression for $V(u,b)$

We look for solutions of (5.6) of the form

$$
V(u,b)=\sum_{l=1}^{n+m} C_l e^{-r_l u} \qquad (5.7)
$$

for some coefficients C_l and some exponents r_l that are up to be determined.

Replacing (5.7) in (5.6) we get

$$
\begin{aligned}
0 &= \left(\left(1+\frac{\delta}{\lambda}\right)I+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n V(u,b)-W(u,b) \\
&= \sum_{l=1}^{n+m} C_l\left[\left(1+\frac{\delta}{\lambda}-\left(\frac{c}{\lambda}\right)r_l\right)^n-\hat{p}(r_l)\right]e^{-r_l u}+ \\
&- \overrightarrow{\alpha}\left[\sum_{l=1}^{n+m} C_l e^{-r_l b}\left((r_l I_m-B)^{-1}B+I_m\right)-B^{-1}\right]e^{B(b-u)}\overrightarrow{\gamma}^t. \qquad (5.8)
\end{aligned}
$$

Since equation (5.8) holds $\forall u$, the coefficients of $e^{-r_l u}$ and $e^{B(b-u)}$ must be zero. This means that

$$
\left(1+\frac{\delta}{\lambda}-\left(\frac{c}{\lambda}\right)r_l\right)^n-\hat{p}(r_l)=0, \quad l=1,\ldots,n+m,
$$

so the exponents r_l are all the $m+n$ roots of the generalized Lundberg's equation (2.2), that in this case will have n roots with positive real parts, namely $r_1, r_2, \ldots, r_n$, and m roots with negative real parts, $r_{n+1}, r_{n+2}, \ldots, r_{n+m}$.

Also,

$$\overrightarrow{\alpha}\left[\sum_{l=1}^{n+m} C_l e^{-r_l b}\left((r_l I_m - B)^{-1}B + I_m\right) - B^{-1}\right] = \overrightarrow{0}. \tag{5.9}$$

This gives a homogeneous system of m equations on the $m+n$ unknown coefficients C_l. The remaining n equations that we need (to have a full system of $m+n$ equations on the $m+n$ unknowns), are the n boundary conditions (5.4).

Example 5.2 *Lets assume that the times between jumps are Erlang($2,\lambda$) distributed and the jump amounts are Erlang($2,\beta$) distributed.*

Then the negative loading condition is $c < \frac{\lambda}{\beta}$ and the generalized Lundberg's equation (2.2) *becomes*

$$(\lambda + \beta - cs)^2(\beta + s)^2 = \lambda^2\beta^2 \tag{5.10}$$

Let

$$V(u,b) = \sum_{l=1}^{4} C_l e^{-r_l u}$$

Then the exponents r_l's are the four roots of (5.10).

From the two boundary conditions (5.4) *we get*

$$\sum_{l=1}^{4} C_l = 0, \quad \sum_{l=1}^{4} C_l r_l = 0,$$

and from (5.9) *we obtain*

$$\sum_{l=1}^{4} C_l e^{-r_l b}\frac{r_l}{r_l+\beta} = -\frac{1}{\beta}, \quad \sum_{l=1}^{4} C_l e^{-r_l b}\frac{r_l\beta}{(r_l+\beta)^2} = -\frac{1}{\beta},$$

so we have a system of four equations in the four unknowns $C_1,\ldots,C_4$. In matrix form

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ r_1 & r_2 & r_3 & r_4 \\ e^{-r_1 b}\frac{r_1}{r_1+\beta} & e^{-r_2 b}\frac{r_2}{r_2+\beta} & e^{-r_3 b}\frac{r_3}{r_3+\beta} & e^{-r_4 b}\frac{r_4}{r_4+\beta} \\ e^{-r_1 b}\frac{r_1\beta}{(r_1+\beta)^2} & e^{-r_2 b}\frac{r_2\beta}{(r_2+\beta)^2} & e^{-r_3 b}\frac{r_3\beta}{(r_3+\beta)^2} & e^{-r_4 b}\frac{r_4\beta}{(r_4+\beta)^2} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{\beta} \\ -\frac{1}{\beta} \end{pmatrix}$$

Now set the values for the parameters $\lambda = \beta = 1$, $c = 0.75$, $\delta = 0.02$. Then $r_1 = 0.423$, $r_2 = 1.831$, $r_3 = -0.063$ and $r_4 = -1.471$. After computing the coefficients we obtain the following table for $u \in \{1,3,5,10,15,20\}$ and $b \in \{2,3,6,10,30,40\}$ The values are quite

$u\backslash b$	2	3	6	10	30	40
1	1.049	1.3	1.856	1.78	0.526	0.28
3		4.533	6.45	6.189	1.826	0.972
5			9.374	8.993	2.653	1.412
10				13.829	4.081	2.172
15					5.647	3.006
20					7.746	4.123

Table 5.1: Values of $V(u,b)$ for different values of u and b

similar to those in table 7.1 of Afonso et al (2011) although a little bit smaller. Also we notice that for a fixed u the value of $V(u,b)$ increases until a certain value of b and then decreases rapidly. This behavior is expected and corroborates the findings of Afonso et al (2011) and Avanzi et al (2007).

6 Concluding remarks

We have found the expressions for the ruin probability and the Laplace transform of the time of ruin in an Erlang(n) dual risk model. Also, we have found a method to compute the expected discounted dividends in an Erlang(n) dual risk model with PH(m) distributed jump amounts. For all the theoretical results we provided explicit examples and numerical computations to compare, whenever possible, with existing results in the literature.

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References

Afonso, L. B., Cardoso R. M. R. and Egídio dos Reis, A. D. (2011). Dividend problems in the dual risk model.

Albrecher, H., Badescu, A. and Landriault, D. (2008). On the dual risk model with tax payments, *Insurance: Mathematics and Economics* **42(3)**, 1086–1094.

Avanzi, B., Gerber, H. U. and Shiu, E. S. W. (2007). Optimal dividends in the dual model, *Insurance: Mathematics and Economics* **41(1)**, 111–123.

Dickson, D. C. M. (2005). *Insurance risk and ruin*, Cambridge University Press.

Gerber, H. U. (1979). *An Introduction to Mathematical Risk Theory*, Huebner Foundation for Insurance Education, S. S.

Ji, L. and Zhang C. (2011). Analysis of the multiple roots of the Lundberg fundamental equation in the PH(n) risk model, *Applied Stochastic Models in Business and Industry*.

Li, S. and Garrido, J. (2004). On ruin for the erlang(n) risk process, *Insurance: Mathematics and Economics* **34**(3), 391–408.

Ng, A. C. Y. (2009). On a dual model with a dividend threshold, *Insurance: Mathematics and Economics* **44(2)**, 315–324.