Moments of Dividends and Optimal Expected Dividends in the Erlang(n) dual risk model

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Abstract

The dual risk model assumes that the surplus of a company decreases at a constant rate over time, and grows by means of upward jumps which occur at random times and sizes. It has applications to companies with economical activities involved in research and development. This model is *dual* to the well known Cramér-Lundberg risk model with applications to insurance.

Existing results on the study of the dual model mostly assume that the random waiting times between consecutive gains follow an exponential distribution, as in the classical Cramér–Lunderg risk model.

We generalize to other compound renewal risk models where such waiting times are Erlang(n) distributed. In a previous work, Rodríguez *et al*. (2012), using the roots of the fundamental and the generalized Lundberg's equation, performed calculations to obtain expressions for the ruin probability and the Laplace transform of the time of ruin for an arbitrary single gain distribution. For the computation of the expected discounted dividends they assumed that the gains follow a Phase–Type, PH(n), distribution.

In the present work we keep the assumption that the gains follow a Phase–Type distribution, and extend the computation to higher moments for the discounted dividends. We also develop a numerical method to determine the dividend barrier that maximizes the expected discounted dividends. Finally, we perform illustrations working some examples for some particular gain distributions and obtain numerical results.

Keywords: dual risk model; Erlang(n) interarrival times; Phase–Type distribution, generalized Lundberg's equation; ruin probability; time of ruin; expected discounted dividends.

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1 Introduction

We consider the dual risk model where the surplus or equity of the company is commonly described as

$$U(t) = u - ct + \sum_{i=0}^{N(t)} X_i,\ t \geq 0,\ u \geq 0,$$

where $X_0 \equiv 0$, u is the initial surplus and c is a constant meaning the rate of expenses, assumed deterministic and fixed. The gains sequence is denoted by $\{X_i\}_{i=0}^{\infty}$, it is a sequence of i.i.d. random variables with common cumulative distribution and density functions $P(x)$, $P(0) = 0$, and $p(x)$, respectively. We assume the existence of $\mu_1 = E[X_1]$. Denote the Laplace transform of $p(x)$ by $\hat{p}(s)$. The model is called *dual* as opposed to the Cramér–Lundberg risk model with applications to insurance. For simplicity we will refer this latter as the *primal* model (where c is the income premium rate and X_i means the i-th single loss).

By $N(t) = \max\{k : T_0+T_1+\cdots+T_k \leq t, T_0 = 0\}$ we denote the number of gains occurring before a given time t, where the random variable T_i, denotes interarrival time between jumps $i-1$ and i (≥ 1). We assume that $\{T_i\}_{i=0}^{\infty}$is a sequence of i.i.d. random variables and also independent from $\{X_i\}$. We further assume that T_1 follows an Erlang(n,λ) distribution, which density is denoted $k_n(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$, $t \geq 0$, $\lambda > 0$, $n \in \mathbb{N}^+$. The corresponding distribution function is denoted as $K_n(t)$.

We assume the existence of the negative loading condition, i.e. $cE(W_1) < E(X_1) \Leftrightarrow cn < \lambda\mu_1$, designated *negative* as opposed to the well known *positive loading* in the classical Sparre–Andersen risk model, where the condition is reversed. For our case the condition means that on average our gains are superior to the expenses, per unit time.

This model has been of great interest in ruin theory in recent times. There are many possible interpretations for the model. We can look at the surplus as the amount of capital of a business engaged in research and development, where gains are random, at random instants, and costs are certain. More precisely, the company pays expenses which occur continuously along time for the research activity and gets occasional revenues according to an Erlang(n,λ) distribution. Revenues can be interpreted as values of future gains from an invention or discovery, the decrease of surplus can represent costs of production, payments to employees, maintenance of equipment, etc.

Among pioneer works on the subject we can cite Cramér (1955), Gerber (1979), Seal (1969), Tákacs (1967). Recent works include those by Avanzi *et al.* (2007), Cheung and Drekic (2008), Ng (2009), Afonso *et al.* (2011).

We introduce an upper barrier into the model and let b denote its level. In the application the barrier means a dividend payment level whose i-th single amont is going to be denoted by the random variable D_i as follows. Each time the surplus process upcrosses level b the excess gain is paid out immediately to the capital holder as a dividend, prior to ruin. Let $\{D_i\}_{i=1}^{\infty}$ be the sequence of the dividend payments and let $D(u,b)$ be the aggregate discounted dividends, at force of interest δ and from initial surplus u. We denote by $V_k(u,b) = E[D(u,b)^k]$, $k \geq 1$, the k-th order moment of $D(u,b)$, for simplicity denote $V(u,b) = V_1(u,b)$.

2 Lundberg's equations

For the Erlang(n,λ) dual risk model, the fundamental Lundberg's equation is

$$\left(1 - \left(\frac{c}{\lambda}\right) s\right)^n = \hat{p}(s), \tag{2.1}$$

and the generalized Lundberg's equation, considering the force of interest $\delta > 0$, is

$$\left(1 + \frac{\delta}{\lambda} - \left(\frac{c}{\lambda}\right) s\right)^n = \hat{p}(s). \tag{2.2}$$

We can think of (2.1) as the limit case of (2.2) when $\delta \to 0^+$.

In Rodríguez *et al.* (2013) we prove that both equations have exactly n roots with positive real parts. Let $\rho_1, \ldots, \rho_n$ denote these roots.

Following Ji and Zhang (2012) we note that roots $\rho_1, \ldots, \rho_n$ are all distinct for $\delta \geq 0$, see end of their Section 1, p. 75. This remark was originally described for the primal model, but it remains valid in the case of the dual (their equation corresponding to (2.2) although dependent of c is irrespective of the loading condition). This feature will be very important later on this manuscript.

3 Expected Discounted Dividends

In Rodríguez *et al.* (2013) we proved the expected discounted dividends $V(u,b)$ satisfies the integro–differential equation

$$\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^{n} V(u,b)=W_{\delta}(u,b), \tag{3.1}$$

with boundary conditions

$$\left.\frac{d^i}{du^i}V(u,b)\right|_{u=0}=0, \quad i=0,\ldots,n-1, \tag{3.2}$$

where

$$W_{\delta}(u,b)=\int_0^{b-u}V(u+y,b)p(y)dy+\int_{b-u}^{\infty}(y+u-b+V(b,b))p(y)dy.$$

Then we worked on the particular case when the single gains follow a distribution of the Phase-Type family, PH(m). Denote by $\mathbf{B}=(b_{ij})_{1\le i,j\le m}$ the matrix of the transition rates between the transient states, let $\alpha'=(\alpha_1,\alpha_2,\ldots,\alpha_m)$ be the vector of the initial probabilities, $\eta'=(\eta_1,\eta_2,\ldots,\eta_m)$ the vector of the exit rates to the absorbing state, and the $1\times m$ vector $\mathbf{1}'=(1,1,\ldots,1)$. Let $\mathbf{I_m}$ denote the identity matrix of order m. It is well known for this family that the probability and distribution functions are denoted as $p(x)=\alpha'^{\mathbf{B}x}\eta$ and that $P(x)=1-\alpha'^{\mathbf{B}x}\mathbf{1}$, respectively.

Using the annihilator method from the theory of ordinary differential equations we found an expression for the expected discounted dividends of the form

$$V(u,b)=\sum_{k=1}^{n+m}a_k e^{-r_k u} \tag{3.3}$$

where the exponents r_l, $l=1,...,n+m$, are all the $m+n$ roots of the generalized Lundberg's equation (2.2), from those n roots have positive real parts, namely $\rho_1,\rho_2,\ldots,\rho_n$, and the remaining m roots have negative real parts, $\rho_{n+1},\rho_{n+2},\ldots,\rho_{n+m}$.

The coefficients a_k are obtained using the boundary conditions (3.2) together with the following condition

$$\alpha' \left[\sum_{l=1}^{n+m} a_l e^{-r_l b} \left((r_l \mathbf{I_m} - \mathbf{B})^{-1} \mathbf{B} + \mathbf{I_m} \right) - \mathbf{B}^{-1} \right] = \mathbf{0}. \tag{3.4}$$

which gives a homogeneous system of $m+n$ equations on the $m+n$ unknowns a_k.

Example 3.1 *In Rodríguez et al. (2013) we gave the following example to obtain values of $V(u,b)$ for a choice of parameters.*

Set the values for the parameters $\lambda = \beta = 1$, $c = 0.75$, $\delta = 0.02$. Then $\rho_1 = 0.423$, $\rho_2 = 1.831$, $\rho_3 = -0.063$ and $\rho_4 = -1.471$. After computing the coefficients we obtain the values of the expected discounted dividends, for $u \in \{1,3,5,10,15,20\}$ and $b \in \{2,3,6,10,30,40\}$, that are shown in Table 3.1. The values are quite similar to those in Table 7.1 of Afonso et al. (2011) although a little bit smaller. Also we notice that for a fixed u the value of $V(u,b)$ increases until a certain value of b and then decreases rapidly. This behavior is expected and corroborates the findings of Afonso et al. (2011) and Avanzi et al. (2007).

$u \backslash b$	2	3	6	10	30	40
1	1.049	1.301	1.856	1.781	0.526	0.279
3		4.533	6.451	6.189	1.826	0.972
5			9.374	8.993	2.653	1.412
10				13.829	4.081	2.172
15					5.647	3.006
20					7.746	4.123

Table 3.1: Expected discounted dividends

3.1 Higher moments of the discounted dividends

In the Erlang(n,λ) model, the k-th ordinary moment of the discounted dividends $V_k(u,b)$ satisfies the renewal equation

$$\begin{aligned} V_k(u,b) &= \int_0^{\frac{u}{c}} k_n(t)e^{-k(\delta)t}\left[\int_0^{b-u+ct} V_k(u-ct+y,b)p(y)dy + \right. \\ & \left. \int_{b-u+ct}^{\infty} \widetilde{V}_k(u-ct+y,b)p(y)dy\right] dt, \end{aligned}$$

with

$$\widetilde{V}_k(x,b) = \sum_{j=0}^{k} \binom{k}{j}(x-b)^j V_{k-j}(b,b), \quad x \geq b.$$

In the above expression we assume the convention $V_0(u,b) \equiv 1$. The corresponding integro–differential equation is

$$\left(\left(1+\frac{k\delta}{\lambda}\right)\mathcal{I} + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n V_k(u,b) = W_{k\delta}(u,b), \tag{3.5}$$

with boundary conditions

$$\left.\frac{d^i}{du^i}V_k(u,b)\right|_{u=0} = 0, \quad i=0,\dots,n-1, \tag{3.6}$$

where

$$W_{k\delta}(u,b) = \int_u^b V_k(x,b)p(x-u)dx + \int_b^{\infty} \widetilde{V_k}(x,b)p(x-u)dx.$$

Assuming that the gains follow a PH(m) distribution we can apply an analogous method to find and an expression for $V_k(u,b)$, and numerical values in the same way. As we did for the case $k=1$ in Rodríguez *et al.* (2013) we apply the annihilator

$$q_{\mathbf{B}}(-\frac{d}{du}) = det(\mathbf{B}+\mathbf{I_m}\frac{d}{du}) = \sum_{i=0}^{m} q_i \frac{d^i}{du^i},$$

to the integro–differential equation (3.5) to obtain

$$
\begin{aligned}
0 &= \sum_{l=0}^{n+m}\left[\sum_{i+j=l} q_i \binom{n}{n-j}\left(1+\frac{k\delta}{\lambda}\right)^{n-j}\left(\frac{c}{\lambda}\right)^j\right]\frac{d^l}{du^l}V_k(u,b)+ \\
&\sum_{j=0}^{m-1}\left[\sum_{i=j+1}^{m} q_i\alpha'(-\mathbf{B})^{i-j}\mathbf{1}\right]\frac{d^j}{du^j}V_k(u,b).. \qquad (3.7)
\end{aligned}
$$

Therefore we seek for solutions of (3.7) of the form

$$
V_k(u,b)=\sum_{l=1}^{n+m} a_l e^{-r_l u}. \qquad (3.8)
$$

Replacing (3.8) in (3.7) gives

$$
\begin{aligned}
0 &= \left(\left(1+\frac{k\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n V(u,b)-W(u,b) \\
&= \sum_{l=1}^{n+m} a_l\left[\left(1+\frac{k\delta}{\lambda}-\left(\frac{c}{\lambda}\right)r_l\right)^n-\hat{p}(r_l)\right]e^{-r_l u} \\
&\quad -\alpha'\left[\sum_{l=1}^{n+m} a_l e^{-r_l b}\left((r_l\mathbf{I_m}-\mathbf{B})^{-1}\mathbf{B}+\mathbf{I_m}\right)\right. \\
&\quad \left.+\sum_{j=1}^{k} j\binom{k}{j}V_{k-j}(b,b)(-\mathbf{B})^{-j}\right]e^{\mathbf{B}(b-u)}\mathbf{1}. \qquad (3.9)
\end{aligned}
$$

Since (3.9) is valid $\forall u$ then we must have

$$
\left(1+\frac{k\delta}{\lambda}-\left(\frac{c}{\lambda}\right)r_l\right)^n-\hat{p}(r_l)=0, \quad l=1,\ldots,n+m,
$$

so the exponents r_l, $l=1,...,n+m$, are all the $m+n$ roots of the generalized Lundberg's equation (2.2), from those n roots have positive real parts, namely $\rho_1,\rho_2,\ldots,\rho_n$, and m roots with negative real parts, $\rho_{n+1},\rho_{n+2},\ldots,\rho_{n+m}$. Also, we must have

$$
\alpha'\left[\sum_{l=1}^{n+m} a_l e^{-r_l b}\left((r_l\mathbf{I_m}-\mathbf{B})^{-1}\mathbf{B}+\mathbf{I_m}\right)+\sum_{j=1}^{k} j\binom{k}{j}V_{k-j}(b,b)(-\mathbf{B})^{-j}\right]=\mathbf{0}. \qquad (3.10)
$$

This last condition, together with the boundary conditions (3.6) gives a homogeneous system of $m+n$ equations on the $m+n$ unknowns a_k.

Example 3.2 *We want to compute $V_2(u,b)$. Lets assume that the time between two consecutive jumps are Erlang(2, λ) distributed and the jump amounts are Erlang(2, β) distributed. Then, the negative loading condition is $c < \lambda/\beta$ and the generalized Lundberg's equation* (2.2) *is given by*

$$(\lambda + k\delta - cs)^2(\beta + s)^2 = \lambda^2\beta^2. \tag{3.11}$$

Let

$$V_2(u,b) = \sum_{l=1}^{4} a_l e^{-\rho_l u}$$

Then the exponents ρ_l are the four roots of (3.11)*. Say that ρ_1, ρ_2 are the two roots with positive real parts and ρ_3, ρ_4 are the ones with negative real parts. From the two boundary conditions* (3.6) *we get*

$$\sum_{l=1}^{4} a_l = 0, \ \text{and} \ \sum_{l=1}^{4} a_l\rho_l = 0,$$

and from (3.10) *we get*

$$\sum_{l=1}^{4} a_l e^{-\rho_l b}\frac{\rho_l}{\rho_l+\beta} = -2V(b,b)\frac{1}{\beta} - \frac{2}{\beta^2}, \quad \sum_{l=1}^{4} a_l e^{-\rho_l b}\frac{\rho_l\beta}{(\rho_l+\beta)^2} = -2V(b,b)\frac{1}{\beta} - \frac{4}{\beta^2},$$

so we have a system of four equations in the four unknowns $a_1, \ldots, a_4$.

In matrix form we have

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \rho_1 & \rho_2 & \rho_3 & \rho_4 \\ e^{-\rho_1 b}\frac{\rho_1}{\rho_1+\beta} & e^{-\rho_2 b}\frac{\rho_2}{\rho_2+\beta} & e^{-\rho_3 b}\frac{\rho_3}{\rho_3+\beta} & e^{-\rho_4 b}\frac{\rho_4}{\rho_4+\beta} \\ e^{-\rho_1 b}\frac{\rho_1\beta}{(\rho_1+\beta)^2} & e^{-\rho_2 b}\frac{\rho_2\beta}{(\rho_2+\beta)^2} & e^{-\rho_3 b}\frac{\rho_3\beta}{(\rho_3+\beta)^2} & e^{-\rho_4 b}\frac{\rho_4\beta}{(\rho_4+\beta)^2} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ -2V(b,b)\frac{1}{\beta} - \frac{2}{\beta^2} \\ -2V(b,b)\frac{1}{\beta} - \frac{4}{\beta^2} \end{pmatrix}.$$

Set values for the parameters as $\lambda = \beta = 1$, $c = 0.75$, $\delta = 0.02$. Then $\rho_1 = 0.494$, $\rho_2 = 1.853$, $\rho_3 = -0.107$ and $\rho_4 = -1.467$. After computing the coefficients we obtain the values *for the standard deviation of $D(u,b)$, for $u \in \{1,3,5,10,15,20\}$ and $b \in \{2,3,6,10,30,40\}$, that are shown in Table 3.2.*

$u \backslash b$	2	3	6	10	30	40
1	2.534	3.389	4.893	4.638	1.655	0.973
3		5.058	7.335	6.906	2.667	1.621
5			7.483	6.985	2.966	1.841
10				6.864	3.531	2.277
15					4.269	2.829
20					5.093	3.496

Table 3.2: Standard deviation of the discounted dividends

4 Optimal Dividend Barrier

For a given initial capital u, let b^* denote the optimal value of the barrier b that maximizes the expected discounted dividends $V(u,b)$. Avanzi *et al.* (2007) shows that for a dual model with exponentially distributed gain interarrival times the value of b^* is independent of u.

We have observed that the same situation occur for a dual model with Erlang(n) distributed gain interarrival times and Phase–Type(m) distributed gain amounts. Let b_k^* be the value that maximizes $V_k(u,b)$, for $k \geq 1$.

Theorem 4.1 *b_k^* is independent of the initial surplus u.*

Example 4.1 *In the examples given in the previous sections, the value of b that maximizes $V(u,b)$ is $b^* = 7,33$, and the corresponding value for $V_2(u,b)$ is $b_2^* = 7,12$.*

Consider the case $k = 1$. Since $V(u,b)$ is maximal at $b = b^*$ for each u, we have

$$\left. \frac{\partial V(u,b)}{\partial b} \right|_{b=b^*} = 0.$$

For the optimal value b^* the smooth pasting condition must be satisfied. For a dual model with Erlang(n) distributed interclaim times this condition states that

$$\begin{aligned}\frac{\partial V(b^*-,b^*)}{\partial u} &= \frac{\partial V(b^*+,b^*)}{\partial u}=1\\ \frac{\partial^k V(b^*-,b^*)}{\partial u^k} &= \frac{\partial^k V(b^*+,b^*)}{\partial u^k}=0, \quad 2\le k\le n,\end{aligned}$$

this means that the left and right partial derivatives have to coincide in the optimal value. Using this we get the following result

$$V(b^*,b^*)=\frac{\lambda^n\mu_1-n(\lambda+\delta)^{n-1}c}{(\lambda+\delta)^n-\lambda^n}.$$

For $n=1$ this agrees with the results of Avanzi *et al.* (2007) and others, where $V(b^*,b^*)=(\lambda\mu_1-c)/\delta$.

References

Afonso, L. B., Cardoso R. M. R. and Egídio dos Reis, A. D. (2011). Dividend problems in the dual risk model.

Albrecher, H., Badescu, A. and Landriault, D. (2008). On the dual risk model with tax payments, *Insurance: Mathematics and Economics* **42(3)**, 1086–1094.

Avanzi, B. (2009). Strategies for dividend distribution: A review, *North American Actuarial Journal* 13(2), 217-251.

Avanzi, B. and Gerber, H.U. (2008). Optimal dividends in the dual model with diffusion, *ASTIN Bulletin*, 38 (2), 653-667.

Avanzi, B., Gerber, H. U. and Shiu, E. S. W. (2007). Optimal dividends in the dual model, *Insurance: Mathematics and Economics* **41(1)**, 111–123.

Bayraktar, E. and Egami, M. (2008). Optimizing venture capital investment in a jump diffusion model, *Mathematical Methods of Operations Research* 67 (1), 21-42.

Bühlmann, H. (1970). *Mathematical Methods in Risk Theory*, Springer Verlag, New York.

Cramér, H. (1955). *Collective Risk Theory: A Survey of the Theory from the Point of View of the Theory of Stochastic Process*, Ab Nordiska Bokhandeln, Stockholm.

Cheung, E.C.K. (2011). A unifying approach to the analysis of business with random gains, *Scandinavian Actuarial Journal*, forthcoming, available from http://www.tandfonline.com/doi/abs/10.1080/03461238.2010.490027.

Cheung, E.C.K. and Drekic, S. (2008). Dividend moments in the dual risk model: exact and approximate approaches, *ASTIN Bulletin* 38 (2), 399-422.

Dickson, D.C.M. and Li, S. (2012). The distributions of some quantities for Erlang(2) risk models, Research Paper Series 226, Centre for Actuarial Studies, The University of Melbourne. `http://www.economics.unimelb.edu.au/ACT/wps2012/Erlang2distributions_final.pdf`

Gerber, H.U. (1979). *An Introduction to Mathematical Risk Theory*, S.S. Huebner Foundation for Insurance Education, University of Pennsylvania, Philadelphia, Pa. 19104, USA.

Gerber, H.U. and Smith, N. (2008). Optimal dividends with incomplete information in the dual model, *Insurance: Mathematics and Economics* 43 (2), 227-233.

Ji, L. and Zhang C. (2012). Analysis of the multiple roots of the Lundberg fundamental equation in the PH(n) risk model, *Applied Stochastic Models in Business and Industry,* **28**(1) 73–90.

Li, S. and Garrido, J. (2004). On ruin for the erlang(n) risk process, *Insurance: Mathematics and Economics* **34**(3), 391–408.

Ng, A.C.Y. (2009). On a dual model with a dividend threshold, *Insurance: Mathematics and Economics* **44(2)**, 315–324.

Ng, A.C.Y. (2010). On the upcrossing and downcrossing probabilities of a dual risk model with phase-type gains, *ASTIN Bulletin* 40 (1), 281-306.

Rodríguez E., Cardoso R. M. R. & Egídio dos Reis, A. D. (2013). Some advances on the Erlang(n) dual risk model, preprint.

Seal, H.L. (1969). *Stochastic Theory of a Risk Business*, Wiley, New York

Song, M., Wu, R. and Zhang, X. (2008). Total duration of negative surplus for the dual model, *Applied Stochastic Models in Business and Industry* 24 (6), 591-600.

Takács, L. (1967). *Combinatorial Methods in the Theory of Stochastic Processes*, Wiley, New York.

Yang, H. and Zhu, J. (2008). Ruin probabilities of a dual Markov-modulated risk model, *Communications in Statistics, Theory and Methods* 37, 3298-3307.

Zill, D.G. (2012). *A First Course in Differential Equations with Modeling Applications*, 10th ed., Cengage Learning.

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