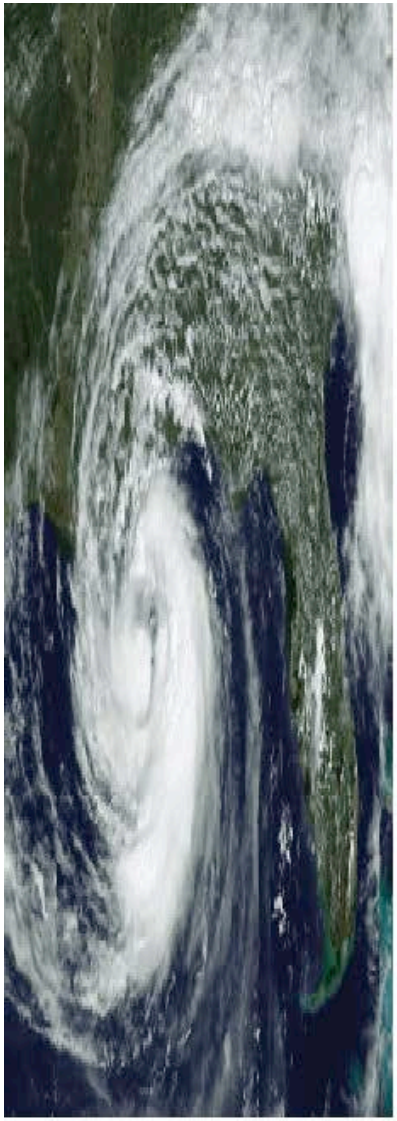
ASTIN, AFIR/ERM and IACA Colloquia
Innovation & Invention

23-27 August 2015 | Amora Hotel Jamison Sydney
Actuaries Institute

Multivariate stop loss mixed Erlang risk: Aggregation, Capital allocation and Default risk

Gildas Ratovomirija

Dependent Large risks





Dependent Large risks

Dependent Large risks

Dependent Large risks

Outline

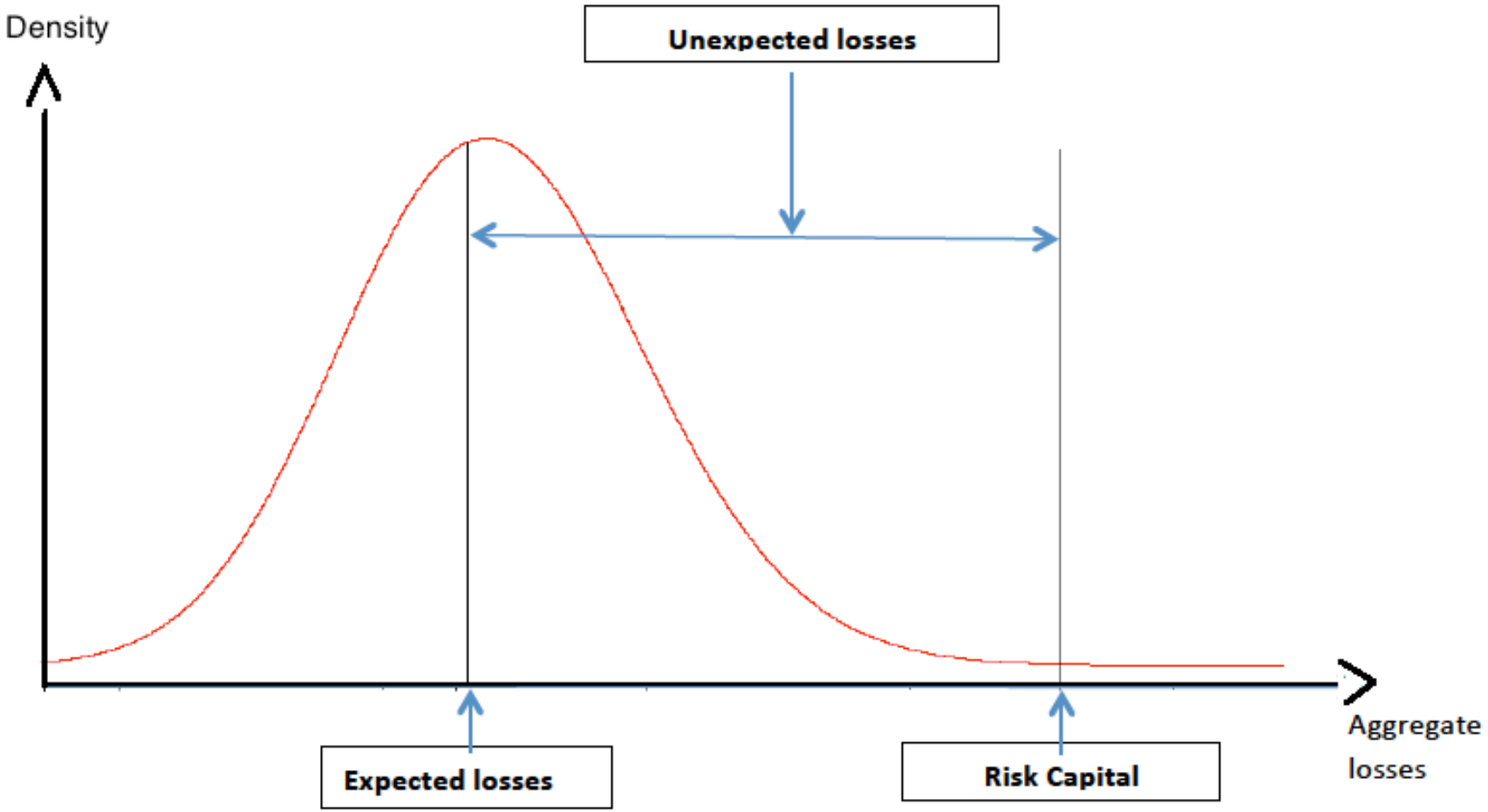
1. Motivations
2. Ceding insurer risk model
3. Reinsurance risk model
4. Numerical illustrations
5. Conclusion

Motivations

- To be covered from **unexpected large losses** insurance companies re-insure some part of their portfolios

- How about these **large risks** in the portfolio of the reinsurer?

Motivations: Risk capital



Motivations: Actuarial goal

Determination of the appropriate **risk capital** with regard to the appropriate dependence structure and the marginal for the mulativariate risk

Motivations: Settings

- X_i: Individual risk of the ceding insurer
- $S_k = \sum_{i=1}^{k} X_i$: Aggregate risk of the ceding insurer
- Y_i: individual risk of the Re-insurer
- $R_k = \sum_{i=1}^{k} Y_i$: aggregate risk of the reinsurer

Motivations: our framework

Ceding insurer risk model:

- Sarmanov distribution as a model of the dependence structure

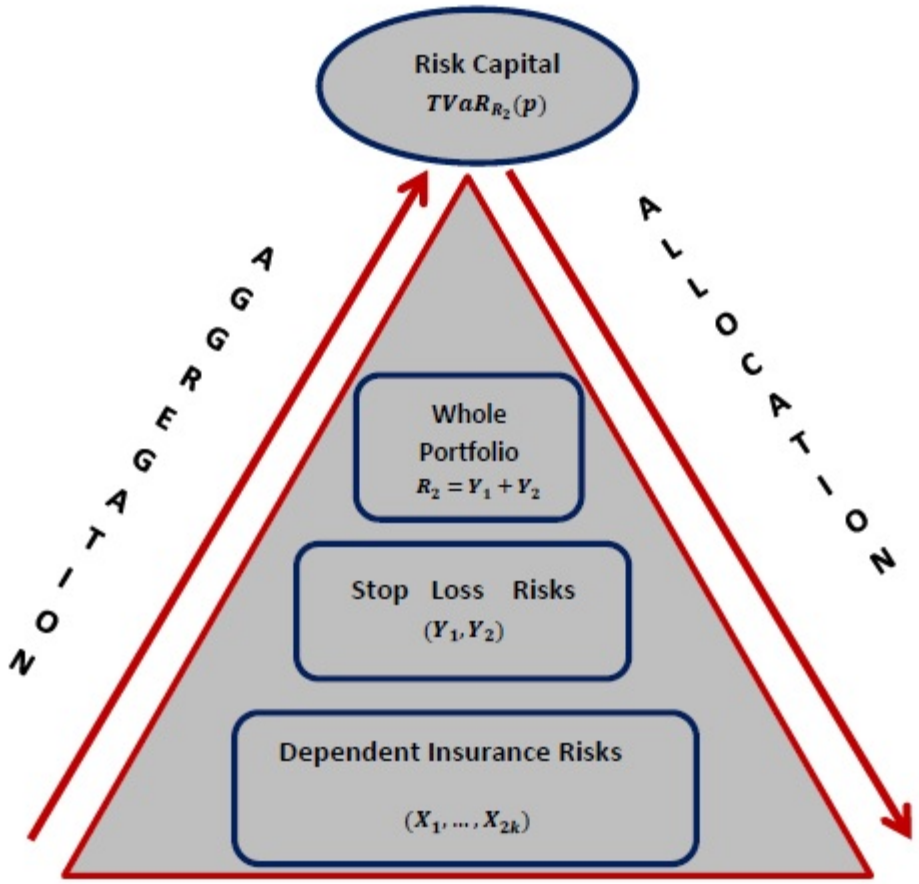
$$(X_1, \ldots, X_k)$$

- X_i: Mixed Erlang distribution with a common scale parameter as marginal

Reinsurer risk model

- $Y_i = (S_k - d_i)_+$: Stop loss reinsurance risk

Motivations: our framework



Outline

1. Motivations

2. Ceding insurer risk model

Bivariate Sarmanov distribution

- The Sarmanov distribution was introduced by Sarmanov (1966)
- The density of a bivariate risk (X_1, X_2) is given by

$$h(x_1, x_2) = f_1(x_1) f_2(x_2) \Big(1 + \alpha_{12} \phi_1(x_1) \phi_2(x_2) \Big), \quad \alpha_{12} \in \mathbb{R}$$

where the two kernel functions ϕ_1, ϕ_2 are chosen such that

$$\mathbb{E}\{\phi_1(X_1)\} = \mathbb{E}\{\phi_2(X_2)\} = 0, \quad 1 + \alpha_{12} \phi_1(x_1) \phi_2(x_2) \geqslant 0$$

Multivariate Sarmanov distribution

- Lee (1996) has introduced the multivariate case
- The density of a multivariate risk $(X_1, \ldots, X_k)$ is given by

$$h(x_1, \ldots, x_k) = \prod_{i=1}^{k} f_i(x_i) \left(1 + \sum_{j<}^{k-1} \sum_{h}^{k} \alpha_{j,h} \phi_j(x_j) \phi_h(x_h) \right), \quad (2.1)$$

such that

$$\mathbb{E}\{\phi_i(X_i)\} = 0,$$

$$1 + \sum_{j<}^{k-1} \sum_{h}^{k} \alpha_{j,h} \phi_j(x_j) \phi_h(x_h) \geqslant 0, \quad \forall x_i \in \mathbb{R}$$

Multivariate Sarmanov distribution

How to choose the kernel function ϕ_i ?

For marginals with suport in $\mathbb{R}_+$:

$\phi_i(x_i) = g_i(x_i) - \mathbb{E}\{g_i(X_i)\}$ as follows, see e.g. Lee (1996):

- $g_i(x_i) = 2\overline{F}_i(xi)$ which corresponds to the FGM distribution
- $g_i(x_i) = x_i^t$ such that $\mathbb{E}\{X_i^t\} < \infty$
- $g_i(x_i) = e^{tx_i}$ such that $\mathbb{E}\{e^{tX_i}\} < \infty$

Sarmanov distribution in insurance applications

- Sarabia and Gómez-Déniz (2011) fitted multivariate insurance count data using Sarmanov distribution with Poisson-Beta marginals
- Hernandez et al. (2012) used the multivariate Sarmanov distribution to model the dependence structure between risk profiles for the calculation of Bayes premium in collective risk model
- By using the Sarmanov distribution, Yang and Hashorva (2013) and Yang and Wang (2013) derived tractable asymptotic formulas in the context of ruin probabilities

Erlang distribution

The Erlang distribution is defined by its shape parameter $k \in \mathbb{N}$ and its scale parameter $\beta > 0$ as follows:

- Pdf

$$w_k(x, \beta) = \frac{\beta^k x^{k-1} e^{-\beta x}}{(k-1)!}$$

- Df

$$W_k(x, \beta) = \sum_{j=k}^{\infty} \frac{(\beta x)^j e^{-\beta x}}{j!},$$

- Survival function

$$\overline{W}_k(x, \beta) = \sum_{j=0}^{k-1} \frac{(\beta x)^j e^{-\beta x}}{j!}, x > 0.$$

Mixed Erlang claim sizes

- $X \sim ME(\beta, \underset{\sim}{V})$ with β the common scale parameter and $\underset{\sim}{V} = (q_1, q_2, \ldots)$ denotes the mixing weights such that $\sum_{k=1}^{\infty} q_k = 1$

- The pdf of X is of the form

$$f(x, \beta, \underset{\sim}{V}) = \sum_{k=1}^{\infty} q_k w_k(x, \beta), \quad x > 0$$

Mixed Erlang claim sizes

Mixed Erlang distribution properties

- Tijms (1994), any positive distribution can be approximated by the mixed Erlang distribution
- The convolution of mixed Erlang risks is again mixed Erlang
- Many useful risk related quantities have closed form, for instance the j-th moment is given by

$$\mu_j = \beta^{-j} \sum_{k=1}^{\infty} q_k \frac{(j+k-1)!}{(k-1)!}.$$

Mixed Erlang claim sizes

Mixed Erlang distribution applications

- In risk theory, Dickson and Willmot (2005) and Dickson (2008) have explored an analytical form of the finite time ruin probability
- Using the EM algorithm, Lee and Lin (2010) have fitted some common parametric distributions and catastrophic loss data in the US with the mixed Erlang distribution
- Lee and Lin (2012) have developped the multivariate Erlang mixtures
- Hashorva and Ratovomirija (2015) have introduced the aggregation of the dependent mixed Erlang risks with Sarmanov distribution

Multivariate SME risks

- Two insurance portfolios, each of which consists of k risks and we denote $S_{1,k} = \sum_{i=1}^{k} X_i$ and $S_{2,k} = \sum_{i=k+1}^{2k} X_i$ the aggregated risk of each portfolio
- $X_i \sim ME(\beta_i, \underset{\sim}{Q}_i)$ with $\underset{\sim}{Q}_i = (q_{i,1}, q_{i,2}, \ldots), i = 1, \ldots, 2k$
- Let $(X_1, \ldots, X_{2k})$ have a multivariate Sarmanov risk with kernel functions, Lee (1996)

$$\phi_i(x_i) = g_i(x_i) - \mathbb{E}\{g_i(X_i)\}$$

- For $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_{2k})$, we denote

$$(X_1, \ldots, X_{2k}) \sim SME(\boldsymbol{\beta}, \underset{\sim}{Q})$$

Correlated insurance portfolios

Proposition 1: Joint tail Probability

If $(X_1, \ldots, X_{2k}) \sim SME(\beta, \underset{\sim}{Q})$ with $\gamma_i := \mathbb{E}\{g_i(X_i)\} < \infty$ and $\beta_{2k} \geq \beta_i, i = 1, \ldots, 2k$, then the joint tail probability of $S_{1,k}$ and $S_{2,k}$ is

$$\mathbb{P}(S_{1,k} > u_1, S_{2,k} > u_2) = \sum_{l=1}^{4} \xi_l \overline{F}_{S_{1,k}^{(l)}}(u_1) \overline{F}_{S_{2,k}^{(l)}}(u_2),$$

Correlated insurance portfolios

Proposition 1: Joint tail Probability

If $(X_1, \ldots, X_{2k}) \sim SME(\boldsymbol{\beta}, \underset{\sim}{Q})$ with $\gamma_i := \mathbb{E}\{g_i(X_i)\} < \infty$ and $\beta_{2k} \geq \beta_i, i = 1, \ldots, 2k$, then the joint tail probability of $S_{1,k}$ and $S_{2,k}$ is

$$\mathbb{P}(S_{1,k} > u_1, S_{2,k} > u_2) = \sum_{l=1}^{4} \xi_l \overline{F}_{S_{1,k}^{(l)}}(u_1) \overline{F}_{S_{2,k}^{(l)}}(u_2),$$

where

$$\xi_1 = 1 + \sum_{j<}^{2k-1} \sum_{h}^{2k} \alpha_{j,h} \gamma_j \gamma_h, \quad \xi_2 = \xi_3 = -\xi_4 = -\sum_{j<}^{2k-1} \sum_{h}^{2k} \alpha_{j,h} \gamma_j \gamma_h,$$

$$S_{1,k}^{(l)} \sim ME(Z(\beta_{2k}), \Pi_k(\underset{\sim}{Q}^{(l)})),$$

$$S_{2,k}^{(l)} \sim ME(Z(\beta_{2k}), \Pi_{2k}(\underset{\sim}{Q}^{(l)})), \quad l=1,2,3,4$$

Outline

1. Motivations

2. Ceding insurer risk model

3. Reinsurance risk model

Independence case

Lemma 1: Stop loss mixed Erlang risk

For a deductible $d > 0$, if $X \sim ME(\beta, \underset{\sim}{V})$ then the df of $Y := (X - d)_+$ is given by

$$F_Y(y) = \begin{cases} F_X(d) & \text{for} \quad y = 0, \\ F_X(d+y) & \text{for} \quad y > 0, \end{cases}$$

where

$$F_X(d+y) = \sum_{k=0}^{\infty} \Delta_k(d, \beta, \underset{\sim}{V}) W_{k+1}(y, \beta),$$

with

$$\Delta_k(d, \beta, \underset{\sim}{V}) = \frac{1}{\beta} \sum_{j=0}^{\infty} q_{j+k+1} w_{j+1}(d, \beta).$$

Independence case

Lemma 2: Aggregation of Stop loss mixed Erlang risks

Let X_1 and X_2 be two independent risks such that $X_i \sim ME(\beta, \underset{\sim}{Q}_i), i = 1, 2$. If $d_i, i = 1, 2$ are positive then $R_2 = Y_1 + Y_2$, with $Y_i = (X_i - d_i)_+, i = 1, 2$, has a df

$$F_{R_2}(s) = \begin{cases} F_{X_1}(d_1)F_{X_2}(d_2) & \text{for} \quad s = 0, \\ F_{X_1}(d_1)F_{X_2}(s + d_2) + F_{X_2}(d_2)F_{X_1}(s + d_1) + F_{X1+X_2}(d_1, d_2, s) & \text{for} \quad s > 0, \end{cases}$$

where

$$F_{X1+X_2}(d_1, d_2, s) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \Delta_k(d_1, \beta, \underset{\sim}{Q}_1)\Delta_j(d_2, \beta, \underset{\sim}{Q}_2)W_{k+j+2}(s, \beta)$$

Independence case

Remark: TVaR of R_2

The TVaR of R_2 at a confidence level $p \in (0,1)$ is given by (set $x_p := VaR_{R_2}(p)$)

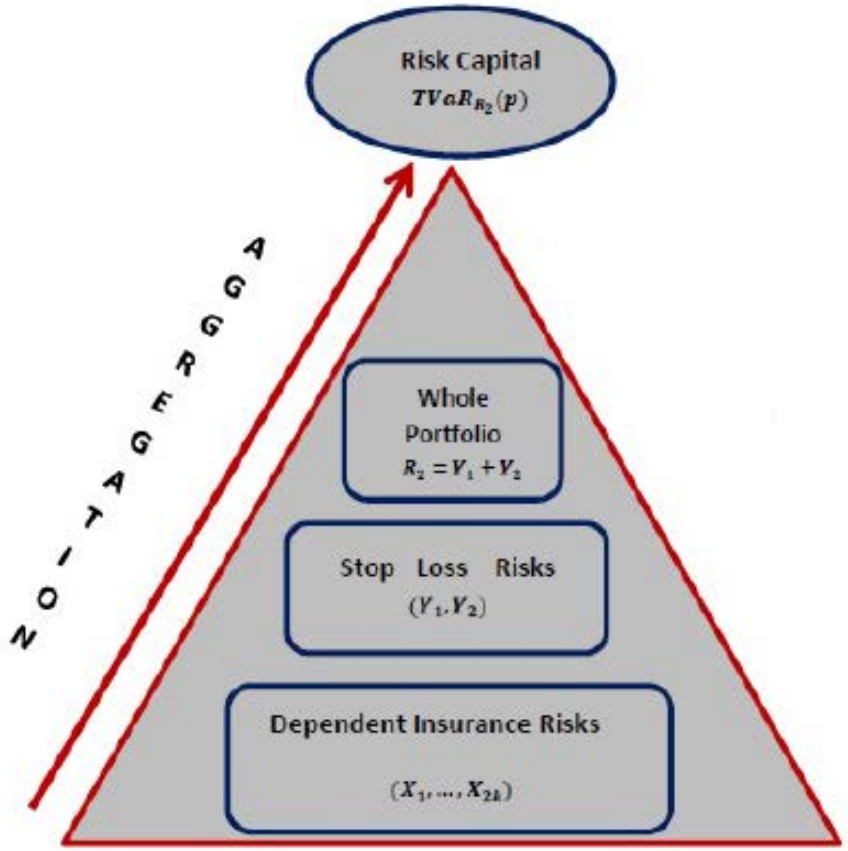
$$TVaR_{R_2}(p) = \frac{1}{1-p}\Big(F_{X_1}(d_1)\overline{U}_{X_2}(x_p, d_2, \beta) + F_{X_2}(d_2)\overline{U}_{X_1}(x_p, d_1, \beta) + \overline{U}_{X_1+X_2}(x_p, d_1, d_2, \beta)\Big),$$

where

$$\overline{U}_{X_i}(x_p, d_i, \beta) = \frac{1}{\beta}\sum_{k=0}^{\infty}(k+1)\Delta_k(d_i, \beta, \underset{\sim}{Q}_i)\overline{W}_{k+2}(x_p, \beta),\ i = 1, 2$$

$$\overline{U}_{X_1+X_2}(x_p, d_1, d_2, \beta) = \frac{1}{\beta}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(k+j+2)\Delta_k(d_1, \beta, \underset{\sim}{Q}_1)\Delta_j(d_2, \beta, \underset{\sim}{Q}_2)\overline{W}_{k+j+3}(x_p, \beta).$$

Dependence case



Dependence case

Proposition 2: Aggregation of Stop loss mixed Erlang risks

If $(X_1, \ldots, X_{2k}) \sim SME(\boldsymbol{\beta}, \underset{\sim}{Q})$ with $\mathbb{E}\{g_i(X_i)\} < \infty$, $\beta_{2k} \geqslant \beta_i, i = 1, \ldots, 2k$ and $d_s > 0, s = 1, 2$, then $R_2 = Y_1 + Y_2$, with $Y_i = (S_{i,k} - d_i)_+, i = 1, 2$, has a df

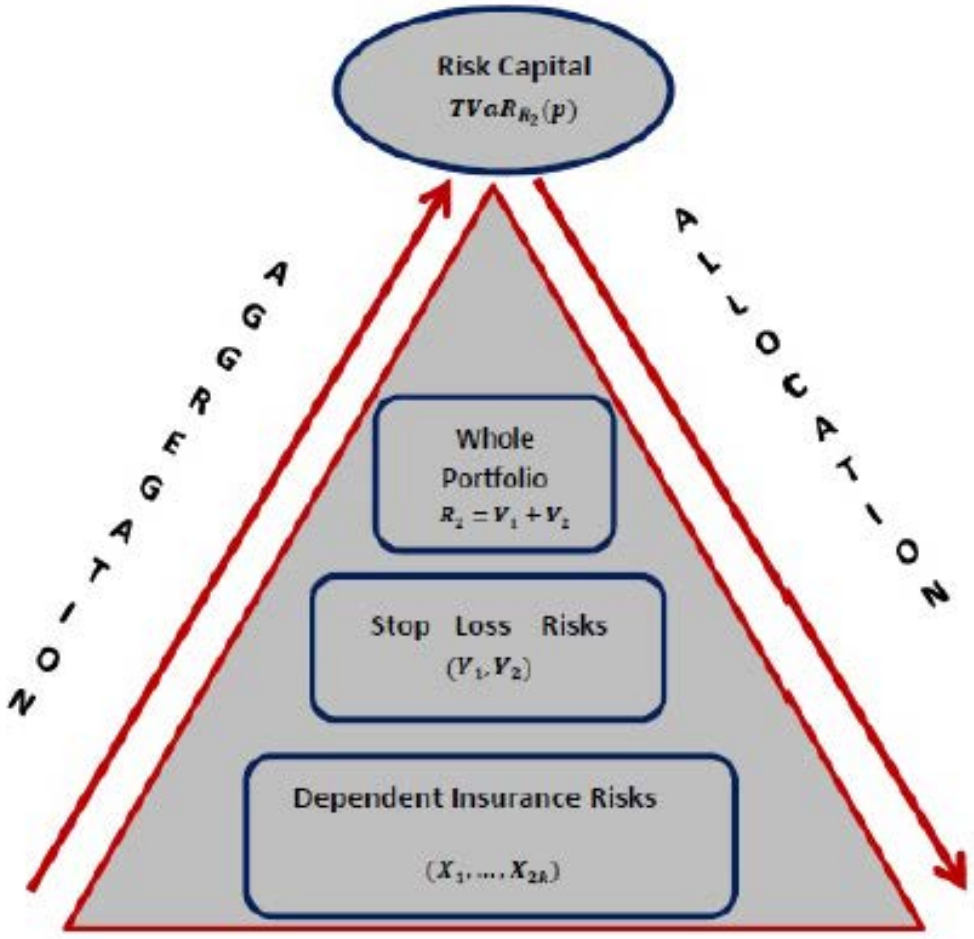
$$F_{R_2}(s) = \begin{cases} F_{S_{1,k},S_{2,k}}(d_1, d_2) & \text{for} \quad s = 0, \\ F_{S_{1,k},S_{2,k}}(d_1 + s, d_2 + s) & \text{for} \quad s > 0, \end{cases}$$

where

$$F_{S_{1,k},S_{2,k}}(d_1, d_2) = \sum_{l=1}^{4} \xi_l F_{S_{1,k}^{(l)}}(d_1) F_{S_{2,k}^{(l)}}(d_2),$$

$$F_{S_{1,k},S_{2,k}}(d_1 + s, d_2 + s) = \sum_{l=1}^{4} \xi_l \Big(F_{S_{1,k}^{(l)}}(d_1) F_{S_{2,k}^{(l)}}(d_2 + s) + F_{S_{1,k}^{(l)}}(d_1 + s) F_{S_{2,k}^{(l)}}(d_2) + F_{S_{1,k}^{(l)} + S_{2,k}^{(l)}}(d_1, d_2, s) \Big).$$

Capital allocation



Capital allocation

Proposition 3: TVaR capital allocation

Let $(X_1, \ldots, X_{2k}) \sim SME(\underset{\sim}{\beta}, \underset{\sim}{Q})$ with $\mathbb{E}\{g_i(X_i)\} < \infty$, $\beta_{2k} \geq \beta_i, i = 1, \ldots, 2k$ and $d_s > 0, s = 1, 2$. If further $Y_i, i = 1, 2$ has finite mean then for a tolerance level $p \in (0, 1)$ the capital allocated to the stop loss risk T_1 under the TVaR allocation principle is given by

$$TVaR_p(Y_1, R_2) = \frac{1}{1-p} \sum_{l=1}^{4} \xi_l \left(F_{S_{2,k}^{(l)}}(d_2) \overline{U}_{S_{1,k}^{(l)}}(x_p, d_1, Z(\beta_{2k})) + \overline{U}_{S_{1,k}^{(l)}}(x_p, d_1, d_2, Z(\beta_{2k})) \right).$$

Outline

1. Motivations
2. Ceding insurer risk model
3. Reinsurance risk model
4. Numerical illustrations

Marginal distribution

Portfolio	X_i	β_i	$\underset{\sim}{Q}_i$	Mean	Variance	Skewness	Kurtosis
A	X_1	0.12	(0.4,0.6)	13.33	127.78	1.55	6.50
	X_2	0.14	(0.3,0.7)	12.14	97.45	1.49	4.33
B	X_3	0.15	(0.5,0.5)	10.00	77.78	1.62	6.80
	X_4	0.16	(0.8,0.2)	7.50	53.13	1.88	8.16

Table: Parameters and central moments of X_i, $i = 1, 2, 3, 4$.

The pdf of X_1, X_2, X_3 and X_4 are repectively given by

$$\begin{aligned} f_1(x_1) &= 0.4w_1(x_1, 0.12) + 0.6w_2(x_1, 0.12), \\ f_2(x_2) &= 0.3w_1(x_2, 0.14) + 0.7w_2(x_2, 0.14), \\ f_3(x_3) &= 0.5w_1(x_3, 0.15) + 0.5w_2(x_3, 0.15), \\ f_4(x_4) &= 0.8w_1(x_4, 0.16) + 0.2w_2(x_4, 0.16) \end{aligned}$$

Dependence parameters

In this example we consider two particular cases

- $g_i(x_i) = 2\overline{F}_i(x_i)$, FGM case, Cossette et al. (2013)
- $g_i(x_i) = e^{x_i}$, Laplace case, Enkelejd and Ratovomirija (2015)

	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,4}$	$\alpha_{2,3}$	$\alpha_{2,4}$	$\alpha_{3,4}$
FGM case	0.6	0.1	0.1	0.1	0.04	0.5
Laplace case	16	5	3	5	3	8

Table: Dependence parameters of (X_1, X_2, X_3, X_4).

- Stop Loss reinsurance deductible for each portfolio: $d_1 = 40$ and $d_2 = 30$

Risk capital and TVaR capital allocation

	FGM case		
p	$TVaR_{R_2}(p)$	$TVaR_p(Y_1, R_2)$	$TVaR_p(Y_2, R_2)$
90.00 %	25.42	16.22	9.20
92.50 %	28.70	18.66	10.04
95.00 %	33.25	22.15	11.10
97.50 %	40.85	28.25	12.60
99.00 %	50.63	36.41	14.22
99.50 %	57.87	42.56	15.31
99.90 %	74.27	56.63	17.64

Table: TVaR and allocated capital to each stop loss risk $Y_i, i = 1, 2,.$

Risk capital and TVaR capital allocation

	Laplace case		
p	$TVaR_{R_2}(p)$	$TVaR_p(T_1, R_2)$	$TVaR_p(T_2, R_2)$
90.00 %	23.10	14.39	8.71
92.50 %	26.24	16.64	9.59
95.00 %	30.60	19.91	10.69
97.50 %	37.93	25.68	12.25
99.00 %	47.43	33.54	13.89
99.50 %	54.50	39.58	14.91
99.90 %	70.58	53.63	16.95

Table: TVaR and allocated capital to each stop loss risk $Y_i, i = 1, 2, .$

$$TVaR_{R_2}(p) = TVaR_p(Y_1, R_2) + TVaR_p(Y_2, R_2)$$

Conclusion

- The mixed Erlang distribution is a tractable marginal for the Sarmanov distribution

Conclusion

- The mixed Erlang distribution is a tractable marginal for the Sarmanov distribution
- The aggregated Sarmanov mixed Erlang risk belongs to the class of Erlang mixtures

Conclusion

- The mixed Erlang distribution is a tractable marginal for the Sarmanov distribution
- The aggregated Sarmanov mixed Erlang risk belongs to the class of Erlang mixtures
- Exact results on the aggregated stop loss reinsurance risk are obtained namely the the risk capital, the capital allocated to each sub portfolio

Conclusion

- The mixed Erlang distribution is a tractable marginal for the Sarmanov distribution
- The aggregated Sarmanov mixed Erlang risk belongs to the class of Erlang mixtures
- Exact results on the aggregated stop loss reinsurance risk are obtained namely the the risk capital, the capital allocated to each sub portfolio
- How about the distribution of

$$(Y_1, \ldots, Y_n) \quad \textit{and} \quad R_n = Y_1 + \ldots + Y_n$$

Thank you for your attention !!!

References

- Hashorva,E. and Ratovomirija,G. (2015). On Sarmanov Mixed Erlang risks in insurance applications. *Astin Bulletin* **45** 175–205.
- Cossette, H., Côté, M.-P., Marceau, E. and Moutanabbir, K. (2013). Multivariate distribution defined with Farlie-Gumbel-Morgenstern copula and mixed Erlang marginals: Aggregation and capital allocation. *Insurance: Mathematics & Economics* **52** 560–572.
- Lee, S. C. K. and Lin, X. S.(2010). Modeling and evaluating insurance losses via mixtures of Erlang distributions. *North American Actuarial Journal* **14**(1)107–130.
- Lee, S. C. K. and Lin, X. S. (2012). Modeling dependent risks with multivariate Erlang mixtures. *Astin Bulletin* **42** 153–180.