Existence and uniqueness of chain ladder solutions

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Introductory remarks

- Existence and uniqueness of chain ladder solutions?
  - Why might they not exist or not be unique?
  - Some chain ladder estimates are the solutions to implicit equations

Framework and notation

- **Array** $\mathfrak{D}$ of **claim observations** $Y_{kj} > 0$:
  - Accident periods are rows $k = 1,2, \ldots, K$
  - Development periods are columns $j = 1,2, \ldots, J$
- Initially, for simplicity, the arrays will be **trapezoidal**:
  - $j = 1,2, \ldots \min(J, K - k +$

Model

- Consider an array subject to the following conditions:
  1) The array $\mathfrak{D}$ is trapezoidal
  2) The random variables $Y_{kj} \in \mathfrak{D}$ are stochastically independent
  3) For each $k = 1,2,\dots,K$ and $j = 1,2,\dots,J$,
     a) Y_{kj} is distributed according to a member of the EDF:

$$\ell_{kj}(y|\mu,\phi) = [yc(\mu_{kj}) - d(\mu_{kj})]/\phi_{kj} + \lambda(y,\phi_{kj})$$

     b) $E[Y_{kj}] = \alpha_k \beta_j$ for some parameters $\alpha_k, \beta_j > 0$
     c) $\sum_{j=1}^{J} \beta_j = 1$

$\boldsymbol{E}[Y_{kj}]$

Require estimation

- This model is called an **EDF cross-classified model**
  - Some times referred to as **ANOVA** instead of cross-classified

Maximum likelihood equations

- ML equations are:

$$\sum_{\mathcal{R}(k)} [Y_{kj} - \mu_{kj}] c'(\mu_{kj}) \beta_j / \phi_{kj} = 0$$

Sum across a whole row

$$\sum_{\mathcal{C}(j)} [Y_{kj} - \mu_{kj}] c'(\mu_{kj}) \alpha_k / \phi_{kj} = 0$$

Sum down a whole column

$$\mu_{kj} = E[Y_{kj}] = \alpha_k \beta_j$$

- These equations are in general implicit
  - Does a solution exist?
  - If so, is it unique?

Special cases

$$\sum_{\mathcal{R}(k)} [Y_{kj} - \mu_{kj}] c'(\mu_{kj}) \beta_j / \phi_{kj} = 0$$

$$\sum_{\mathcal{C}(j)} [Y_{kj} - \mu_{kj}] c'(\mu_{kj}) \alpha_k / \phi_{kj} = 0$$

- Tweedie case:

$$\sum_{\mathcal{R}(k)} [Y_{kj} - \mu_{kj}] \beta_j^{1-p} / \phi_{kj} = 0$$

$$\sum_{\mathcal{C}(j)} [Y_{kj} - \mu_{kj}] \alpha_k^{1-p} / \phi_{kj} = 0$$

- Tweedie family

$$c(\mu) = \mu^{1-p}/(1-p)$$

- Over-dispersed Poisson (**ODP**) family ($p = 1$)

$$c(\mu) = ln\ \mu$$

- ODP case:

$$\sum_{\mathcal{R}(k)} [Y_{kj} - \mu_{kj}] / \phi_{kj} = 0$$

$$\sum_{\mathcal{C}(j)} [Y_{kj} - \mu_{kj}] / \phi_{kj} = 0$$

Explicit solution for the case $\phi_{kj} = \phi$

Existence of solutions

- For the EDF cross-classified model, solutions always exist
  – Proof by application of the Weierstrass theorem

Uniqueness of MLE solution for EDF cross-classified model

- Can be shown that $\boldsymbol{d}'(\boldsymbol{\mu}) = \boldsymbol{\mu}\,\boldsymbol{c}'(\boldsymbol{\mu})$
- Suppose EDF cross-classified model
- Let R denote a compact set $\{\mu_{kj}: 0 < \underline{\mu}_{kj} < \mu_{kj} <$

Uniqueness for special case: Tweedie sub-family

- Suppose $Y_{kj} \sim Tweedie$ with index $p \geq 1$
- Uniqueness condition reduces to:
  - for all $\mu_{kj} \in R$ such that $Y_{kj} \in \mathfrak{D}$,

$$\frac{\boldsymbol{Y_{kj}}}{\boldsymbol{\bar{\mu}_{kj}}} \geq \frac{\boldsymbol{p-2}}{\boldsymbol{p-1}}$$

- Corollary: this condition always satisfied for $\mathbf{1 \leq p \leq 2}$, which includes:
  - ODP ($p = 1$)
  - Gamma ($p = 2$)
  - All compound Poisson with gamma severity ($1 < p < 2$)

Arrays more general than trapezoidal

- Consider an array as an undirected graph $\Gamma(\mathfrak{D})$
  - observations as vertices
  - an edge exists between two observations if and only if they are either from the same row of $\mathfrak{D}$ in adjacent columns, or from the same column of $\mathfrak{D}$ (not necessarily adjacent rows)
- Suppose an array satisfies the following conditions:
  1) Contains a sub-array $\mathcal{S}$ consisting of precisely $K + J - 1$ observations
  2) Each row of $\mathcal{S}$ contains at least one observation, and similarly each column
  3) $\Gamma(\mathcal{S})$ is connected
- Call this array **regular**
- All previous results hold for regular arrays

Uniqueness for Tweedie $(p > 2)$

- Define an array to be **proportional** if $Y_{kj} = \alpha_k \beta_j$
- Define a measure $\xi(\mathfrak{D})$ of non-proportionality of $\mathfrak{D}$
  - Definition as follows for regular array

$$\pi_{rs(j_r,k_s)} = \frac{Y_{rs}}{Y_{rj_r}} \frac{Y_{k_s j_r}}{Y_{k_s s}} - 1 \text{ (provided all quantities involved exist)}$$

$$\bar{\pi}_{rs} = \max_{j_r,k_s} \left[\pi_{rs(j_r,k_s)}\right]_+$$

$$\underline{\pi}_{rs} = \max_{j_r,k_s} [-\pi_{rs}]_+$$

$$\bar{\pi}(\mathfrak{D}) = \max_{r,s} \bar{\pi}_{rs}$$

$$\underline{\pi}(\mathfrak{D}) = \max_{r,s} \underline{\pi}_{rs}$$

$$\xi(\mathfrak{D}) = \frac{1+\bar{\pi}(\mathfrak{D})}{1-\underline{\pi}(\mathfrak{D})} \geq 1 \text{ (=1 for proportional array)}$$

- Sufficient condition for uniqueness when $Y_{kj} \sim Tweedie(p > 2)$

$$p \leq 2 + \frac{1}{\xi(\mathfrak{D})^{J+1}} (\leq 3)$$

Existence of multiple solutions?

- Have obtained sufficient conditions for uniqueness
  - Not necessary conditions
  - Leaves open the question of whether multiple solutions can exist

Numerical example

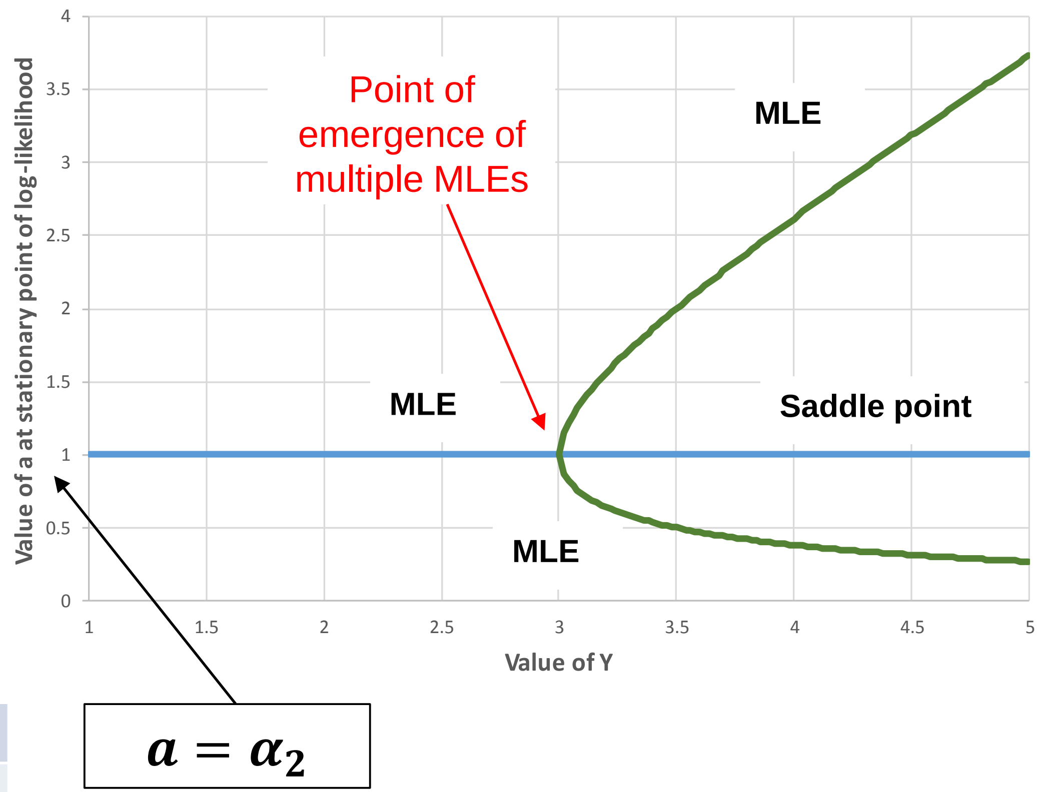
$$\mathfrak{D} = \begin{bmatrix} 1 & Y \\ Y & 1 \end{bmatrix}$$

$Y_{kj} \sim Tweedie(p > 2)$ (inverse Gaussian, $p = 3$)

For $Y = 3.5$ or $Y = {}^{1}/_{3.5}$, three ML solutions:

Value of			
α_1	α_2	β_1	β_2
1	1	2.25	2.25
1	2	1.25	2.5
1	0.5	5	2.5

Loci of stationary points of ℓ with varying Y



Bayesian models

- In Bayesian models, parameters α_k, β_j are randomized
  - ML replaced by MAP estimators
  - Many of the previous results for fixed effect models generalize easily to their Bayesian counterparts

Conclusions

- Under mild regularity conditions, existence always occurs
- For Tweedie error distributions, a unique MLE occurs if:
  - The error distribution is sufficiently well behaved ($1 \leq p \leq 2$); **OR**
  - the data array is sufficiently well behaved (close enough to proportional)
- If both conditions breached, then
  - Behaviour of the likelihood may become complex
  - Multiple solutions can occur

Questions?