Optimal dividend payment under a ruin constraint: discrete time and state space

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Abstract

A modified Hamilton-Jacobi-Bellman (HJB) equation is derived for the problem of optimal dividend payment under a ruin constraint, for discrete time and state space. This equation has a classical solution, and a verification argument is given which shows that the solution is the value function of the problem, and that the maximizer in the HJB equation defines the optimal dividend payment strategy in feedback form.

1 Introduction and Summary

Consider the following stylized model for insurance business: $X_1, X_2, ...$ the total sum of claims per period are iid nonnegative integer valued, c the total premium per period is a positive integer, and the initial surplus s is a non negative integer. The reserve $R(t)$ of the company without dividend payment evolves as $R(0) = s$ and

$$R(t+1) = R(t) + c - X_{t+1}, t \geq 0.$$

Throughout the paper we assume that $P\{X_t > c\} > 0$ and that $c > E[X_t]$; we have a risky insaurance business, and the premium has a positive safety loading.

As a measure of stability we use the infinite time ruin probability

$$\psi^0(s) = P\{R(t) < 0 \text{ for some } t \geq 0\}$$

which satisfies the equation

$$\psi^0(s) = E[\psi^0(s + c - X)].$$

We consider the situation in which dividends $d(t)$ are paid at the beginning of period $t+1, t \geq 0$. If $\mathcal{F}(t)$ is the $\sigma-$field generated by $R(h), h \leq t$, then $d(t)$ is an $\mathcal{F}(t)-$measurable nonnegative random variable. As a measure of profitability we use expected accumulated discounted dividends:

$$u^d(s) = E\left[\sum_{t=0}^{\tau^d-1} v^t d(t)\right].$$

With dividend payment the reserve is $R^d(t)$ defined by $R^d(0) = s$ and

$$R^d(t+1) = R^d(t) - d(t) + c - X_{t+1}, t \geq 0. \tag{1}$$

In the upper index of summation we use τ^d as the ruin time in the risk process $R^d(t)$, i.e.

$$\tau^d = \inf\{t \geq 0 : R^d(t) < 0\},$$

where the infimum is $+\infty$ in case $R^d(t) \geq 0$ for all $t \geq 0$. The ruin probability of the reserve $R^d(t)$ is denoted by

$$\psi^d(s) = P\{\tau^d < \infty\}.$$

There is a tradeoff between stability and profitability: Minimizing ruin probability means no dividend payment, $d(t) \equiv 0$ or $u^d(s) = 0$, and the reserve process $R(t)$ goes to $+\infty$. Maximizing $u^d(s)$ leads to a dividend payment scheme for which ruin is certain,

$$\psi^d(s) = 1 \text{ for all } s \geq 0,$$

and the reserve process $R^d(t)$ remains bounded (see Bühlmann (1996, chapter 6.4) and references given there, as well as Gerber (1979)).

We shall solve the problem of optimal dividend payment under a ruin constraint, i.e. for $0 < \alpha \leq 1$ and initial surplus fixed we shall derive an optimal dividend payment scheme $d(t)$ for which

$$\psi^d(s) \leq \alpha \tag{2}$$

and for which $u^d(s)$ is maximal in the class of all dividend payment schemes satisfying the constraint (2). This is done using a modified Hamilton-Jacobi-Bellman (HJB) equation and via the construction of the process of optimal admissible ruin probabilities.

The HJB equation for the value function $u(s)$ of the problem to maximize profitability is

$$u(s) = \sup_{\delta}\{\delta + vE[u(s - \delta + c - X)]\}, \tag{3}$$

where the supremum is taken over all $0 \leq \delta \leq s$. The optimal strategy is then defined via (1) and

$$d(t) = \delta(R^d(t)),$$

where $\delta = \delta(s)$ is the maximizer in (3).

The modified HJB for the value function $u(s, \alpha)$ under the constraint (2) is

$$u(s, \alpha) = \sup_{\delta, \beta}\{\delta + vE[u(s - \delta + c - X, \beta(X))]\}, \tag{4}$$

where the supremum is taken over all $0 \leq \delta \leq s$ and functions $\beta(x)$ satisfying $E[\beta(X)] \leq \alpha$ and

$$\psi^0(s - \delta + c - x) \leq \beta(x) \leq 1.$$

If there is no admissible pair (δ, β), then the supremum is interpreted as zero. Below we will show that equation (4) has a solution, and that the supremum is

attained at certain values $\delta = \delta(s,\alpha)$ and $\beta(x) = \beta(s,\alpha;x)$. With these values the process of optimal admissible ruin probabilities $b(t), t \geq 0$, is defined as $b(0) = \alpha$ and

$$b(t+1) = \beta(R^d(t), b(t); X_{t+1}), t \geq 0, \tag{5}$$

and the optimal dividend payment strategy is defined through

$$d(t) = \delta(R^d(t), b(t)), t \geq 0. \tag{6}$$

In the next section we show that $d(t)$ is an admissible strategy satisfying the constraint (2), and that $d(t)$ maximizes profitability under this constraint. The process $b(t), t \geq 0$, is a martingale with mean α satisfying

$$b(t+1) \geq \psi^0(R^d(t) - d(t) + c - X_{t+1}) = \psi^0(R^d(t+1)), t \geq 0.$$

The strategy $d(t)$ is path dependent: for $t \geq 0$ the value $d(t)$ depends on $R^d(t-1), b(t)$.

Earlier approaches to optimal dividend payment without constraints or with different constraints can be found in Bühlmann's book (1996), in Gerber (1979 and 1981), and in Paulsen (2003). The Lagrange multiplier method used in Altman (1999) does not seem to work in the infinite horizon situation considered here. Hipp and Schmidli (2003) compute optimal dividend strategies satisfying (2) of the form

$$d(t) = \begin{cases} 0 \text{ if } R^d(t) \leq c(s,\alpha) \\ M \text{ if } R^d(t) > c(s,\alpha) \end{cases} \tag{7}$$

for compound risk processes $R(t)$ in continuous time, and for exponentially distributed claim sizes. These strategies are optimal only in the class of strategies having form (7). They show that optimal strategies within the class of all admissible strategies satisfying (2) can be derived from a modified HJB, adjusted to the Lundberg model:

$$\begin{aligned} 0 = \min[\sup_{\beta}\{&\lambda E[u(s+c-X, \beta(X)) - u(s,\alpha)] - \rho u(s,\alpha) \\ &+ c u_s(s,\alpha) - \lambda(E[\beta(X)] - \alpha) u_\alpha(s,\alpha)\}, 1]. \end{aligned} \tag{8}$$

Again, the supremum is taken over all functions $\beta(x)$ satisfying the following constraint:

$$\beta(x) \geq \psi^0(s-x).$$

2 Statements and proofs

For notational convenience we define $d(t) = 0$ as soon as $t \geq \tau^d$, and that the risk processes are stoppet at τ^d : $R^d(t) = R^d(\tau^d)$ for all $t \geq \tau^d$. We first observe that we may restrict $\delta = \delta(s,\alpha)$ to the set $0, ..., s$.

Lemma 1 *For arbitrary dividend payment strategy $d(t)$ with values in $[0, s]$ there exists a strategy $g(t)$ with $\tau^g = \tau^d$ and*

$$E\left[\sum_{i=1}^{\infty} v^i g(i)\right] \geq E\left[\sum_{i=1}^{\infty} v^i d(i)\right].$$

Proof. Let $D(t) = d(1) + ... + d(t)$ be the accumulated dividend payments, and define $g(t)$ as the increments of $G(t) = [D(t)]$, the smallest integer $\geq D(t)$. Then

$$R^g(t) = [R^d(t)]$$

and hence for integer claims and positive integral initial surplus s we have $\tau^g = \tau^d$. Furthermore, with

$$a_i = v^{i-1} - v^i \geq 0$$

we have

$$\sum_{i=1}^{\infty} v^i d(i) = \sum_{i=1}^{\infty}\sum_{j=i}^{\infty} a_j d(i) =$$

$$\sum_{j=1}^{\infty} a_j \sum_{i=1}^{j} d(i) = \sum_{j=1}^{\infty} a_j D(j) \leq$$

$$\sum_{j=1}^{\infty} a_j [D(j)] = \sum_{i=1}^{\infty} v^i g(i).$$

■

For the existence of a solution to (4) as well as for the numerical calculation we use the functions $u_n(s, \alpha)$ defined recursively as $u_0(s, \alpha) = 0$ and

$$u_{n+1}(s, \alpha) = \sup_{\delta, \beta}\{\delta + vE[u_n(s - \delta + c - X, \beta(X))]\}, \tag{9}$$

where the supremum is taken over all $0 \leq \delta \leq s$ and functions $\beta(x)$ satisfying $E[\beta(X)] \leq \alpha$ and

$$\psi^0(s - \delta + c - x) \leq \beta(x) \leq 1.$$

We define these functions for $s = 0, 1, 2, ...$ and $0 < \alpha \leq 1$.

Lemma 2 *a) For $n \geq 0$ and $0 < \alpha \leq 1$,$s \geq 0$ we have*

$$u_n(s, \alpha) \leq u_{n+1}(s, \alpha) \leq s + \frac{c}{1 - v}.$$

b) There exists a solution to equation (4), and this solution is unique.

c) If s and α are fixed and if $d(t), t \geq 0$, is an arbitrary admissible dividend payment strategy satisfying the constraint (2), then

$$E\left[\sum_{i=0}^{\infty} v^i d(i)\right] \leq u(s, \alpha).$$

d) For $\varepsilon > 0$, $s \geq 0$, and $0 < \alpha \leq 1$ there exists an admissible dividend payment strategy $d(t), t \geq 0$, with corresponding ruin time τ^d satisfying $P\{\tau^d < \infty\} \leq \alpha$ and

$$E\left[\sum_{i=0}^{\infty} v^i d(i)\right] \geq u(s,\alpha) - \varepsilon.$$

e) If $vP\{X < c\} > 1/2$ then for $s \geq 0$ the function $\alpha \to u(s,\alpha)$ is continuous on $(0,1]$.

f) If $vP\{X < c\} > 1/2$ then the supremum δ, β in equation (4) is attained at values $\delta(s,\alpha)$ and $\beta(s,\alpha;x)$. Furthermore,

$$E\left[\sum_{i=0}^{\infty} v^i d(i)\right] = u(s,\alpha).$$

g) The maximizer $\delta = \delta(s,\alpha)$ in (9) satisfies $\delta \leq s(\alpha)$, where $s(\alpha) = \max\{k : \psi^0(k) \geq \alpha\}$.

Proof. a) By induction. The relation $u_0(s,\alpha) \leq u_1(s,\alpha)$ is obvious. Assume now that $u_n(s,\alpha) \geq u_{n-1}(s,\alpha)$. Then

$$\begin{aligned} u_{n+1}(s,\alpha) &= \sup_{\delta,\beta}\{\delta + vE[u_n(s-\delta+c-X,\beta(X))]\} \\ &\geq \sup_{\delta,\beta}\{\delta + vE[u_{n-1}(s-\delta+c-X,\beta(X))]\} \\ &= u_n(s,\alpha). \end{aligned}$$

Furthermore, if $u_n(s,\alpha) \leq s + c/(1-v)$, then

$$\begin{aligned} u_{n+1}(s,\alpha) &= \sup_{\delta,\beta}\{\delta + vE[u_n(s-\delta+c-X,\beta(X))]\} \\ &\leq \sup_{\delta,\beta}\{\delta + v(s-\delta+c-E[X]+c/(1-v))]\} \\ &\leq \sup_{\delta,\beta}\{\delta + v(s-\delta)+c+cv/(1-v))]\} \\ &\leq s + c/(1-v). \end{aligned}$$

b) Since $u_n(s,\alpha)$ is a non decreasing sequence of functions which are bounded from above, there exists a function $u(s,\alpha)$ which is the pointwise limit of $u_n(s,\alpha)$. By dominated convergence, we have for δ, β fixed

$$E[u_n(s-\delta+c-X,\beta(X))] \to E[u(s-\delta+c-X,\beta(X))]$$

and hence

$$\begin{aligned} u(s,\alpha) \geq u_{n+1}(s,\alpha) &= \sup_{\delta,\beta}\{\delta + vE[u_n(s-\delta+c-X,\beta(X))]\} \\ &\geq \delta + vE[u_n(s-\delta+c-X,\beta(X))] \end{aligned}$$

and $n \to \infty$ together yield

$$u(s,\alpha) \geq \sup_{\delta,\beta}\{\delta + vE[u(s-\delta+c-X,\beta(X))]\}.$$

On the other hand, $u_n(s,\alpha) \leq u(s,\alpha)$ implies

$$\begin{aligned}
&\sup_{\delta,\beta}\{\delta + vE[u(s-\delta+c-X,\beta(X))]\} \\
&\geq \sup_{\delta,\beta}\{\delta + vE[u_n(s-\delta+c-X,\beta(X))]\} \\
&= u_n(s,\alpha),
\end{aligned}$$

and the last term converges to $u(s,\alpha)$. Uniqueness of the solution for equation (4) follows from the fact that the operator

$$Tu(s,\alpha) = \sup_{\delta,\beta}\{\delta + vE[u(s-\delta+c-X,\beta(X))]\}$$

is a contraction: for functions $u_1(s,\alpha)$ and $u_2(s,\alpha)$ we have

$$|Tu_1(s,\alpha) - Tu_2(s,\alpha)| \leq v \sup\{|u_1(y,\alpha) - u_2(y,\alpha)| : 0 \leq y \leq s, 0 \leq \alpha \leq 1\}.$$

To see this, choose δ^* and β^* (depending on ε, s, α) such that

$$Tu_1(s,\alpha) \leq \delta^* + vE[u_1(s-\delta+c-X,\beta^*(X))] + \varepsilon.$$

Then

$$\begin{aligned}
Tu_1(s,\alpha) - Tu_2(s,\alpha) &\leq \delta^* + vE[u_1(s-\delta+c-X,\beta^*(X))] \\
&\quad -\delta^* - vE[u_2(s-\delta+c-X,\beta^*(X))] + \varepsilon \\
&\leq v \sup\{|u_1(y,\alpha) - u_2(y,\alpha)| : 0 \leq y \leq s, 0 \leq \alpha \leq 1\} + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary and $u_1(s,\alpha), u_2(s,\alpha)$ can be interchanged, we obtain the above inequality.

c) Let $R^d(t)$ and τ^d be the risk process and the ruin time corresponding to the dividend payment strategy $d(t)$, respectively. For $t = 0, 1, 2, ...$ let

$$b^d(t) = P\{\tau^d < \infty \mid \mathcal{F}(t)\}.$$

The process $b^d(t), t \geq 0$, is a martingale, i.e.

$$E[b^d(t+1) \mid \mathcal{F}(t)] = b^d(t).$$

Since the sigma-field $\mathcal{F}(t+1)$ is generated by $\mathcal{F}(t)$ and $X_{t+1}, b^d(t+1)$ can be written as a function β^d - depending on $\mathcal{F}(t)$ - of x for which

$$E[\beta^d(X)] = b^d(t).$$

The above expectation is taken over the claim size X with fixed values for $R^d(0), ..., R^d(t)$. If $\tau^d > t$ then

$$b^d(t+1) = P\{R^d(h) < 0 \text{ for some } h \geq t+1 |\ \mathcal{F}(t+1)\},$$

and since ruin is more probable with dividend payments,

$$b^d(t+1) \geq \psi^0(R^d(t+1)).$$

So the function $\beta^d(x)$ satisfies

$$\beta^d(x) \geq \psi^0(R^d(t) - d(t) + c - x).$$

Now for $t = 0, 1, 2...$ define

$$F(t) = E\left[\left(\sum_{i=0}^{t} v^i d(i) + v^{t+1} u(R^d(t+1), b^d(t+1))\right) 1_{\{\tau^d > t\}}\right].$$

We show that $F(t)$ is non decreasing, i.e. $F(t) \leq F(t-1)$. Consider

$$\begin{aligned} &E\left[\left(d(t) + vu(R^d(t+1), b^d(t+1))\right) 1_{\{\tau^d > t\}} \mid \mathcal{F}(t)\right] \\ &= d(t) + vE[u(R^d(t) - d(t) + c - X, \beta^d(X))]. \end{aligned}$$

This is not larger than

$$\sup_{\delta,\beta}\{\delta + vE[u(R^d(t) - \delta + c - X, \beta(X))],$$

where the supremum is taken over $0 \leq \delta \leq s$ and functions $\beta(x)$ with $E[\beta(X)] = b^d(t)$ and

$$\beta(x) \geq \psi^0(R^d(t) - \delta + c - x).$$

This supremum equals $u(R^d(t), b^d(t))$, and hence

$$E\left[\left(d(t) + vu(R^d(t+1), b^d(t+1))\right) 1_{\{\tau^d > t\}} \mid \mathcal{F}(t)\right] \leq u(R^d(t), b^d(t)).$$

Inserting this into the definition of $F(t)$ we obtain

$$\begin{aligned} F(t) &= E\left[\left(\sum_{i=0}^{t-1} v^i d(i) + v^t\{d(t) + vu(R^d(t+1), b^d(t+1))\right) 1_{\{\tau^d > t\}}\right] \\ &\leq E\left[\left(\sum_{i=0}^{t-1} v^i d(i) + v^t u(R^d(t), b^d(t))\right) 1_{\{\tau^d > t\}}\right] = F(t-1). \end{aligned}$$

With the same argument we obtain that

$$F(0) \leq u(R^d(0), b^d(0)) = u(s, b^d(0)) \leq u(s, \alpha).$$

This implies the assertion:

$$\begin{aligned} E\left[\sum_{i=0}^{\infty} v^i d(i)\right] &= E\left[\sum_{i=0}^{\infty} v^i d(i) 1_{\{\tau^d = \infty\}}\right] \\ +E\left[\sum_{i=0}^{\infty} v^i d(i) 1_{\{\tau^d < \infty\}}\right] &= \lim_{t\to\infty} F(t), \end{aligned}$$

using the relation $u(s,\alpha)=0$ for $s<0$ on the set $\{\tau^d<\infty\}$ and

$$v^{t+1}u(R^d(t),b^d(t))\to 0$$

on the set $\{\tau^d=\infty\}$.

d) Let $\xi=\varepsilon(1-v)$, and let $\delta(s,\alpha)$ and $\beta(s,\alpha;x)$ be chosen such that $E[\beta(s,\alpha;X)]=\alpha,\ \beta(s,\alpha,x)\geq\psi^0(s-\delta(s,\alpha)+c-x)$, and

$$\delta(s,\alpha)+vE[u(s-\delta(s,\alpha)+c-X,\beta(s,\alpha;X))]\geq u(s,\alpha)-\xi.$$

For s,α fixed, define the process of running ruin probabilities $b^d(t)$ and the dividend payment strategy $d(t)$ as in (5) and (6). Then, with τ^d the corresponding ruin time of the process $R^d(t)$,

$$\begin{aligned}&E[\big(d(t)+vu(R^d(t+1),b^d(t+1))\big)\,1_{\{\tau^d>t\}}]=\\&\delta+vE[u(R^d(t)-\delta+c-X,\beta(X))1_{\{\tau^d>t\}}]=:I,\end{aligned}$$

where $\delta=\delta(R^d(t),b^d(t))$ and $\beta(x)=\beta(R^d(t),b^d(t);x)$. From

$$I\geq u(R^d(t),b^d(t))-\xi$$

we obtain that the functions $F(t)$ defined above (but with a different strategy $d(t)$) satisfy $F(t)\geq F(t-1)-\xi v^{t-1}$ and hence

$$E\left[\sum_{i=0}^{\infty}v^i d(i)\right]\geq F(0)-\xi/(1-v)=u(s,\alpha)-\varepsilon.$$

This proves the statement in d).

For the proof of e) we first show that under the assumption $vP(X<c)>1/2$ the following holds: for arbitrary dividend payment strategy $d(t)$ with $P\{\tau^d<\infty\}>\psi^0(s)$ we can find a strategy $f(t)$ with

$$P\{b^f(t)>\psi^0(R^f(t))\}>0,\ t=0,1,2,...$$

satisfying

$$E\left[\sum_{i=0}^{\infty}v^i d(i)\right]\leq E\left[\sum_{i=0}^{\infty}v^i f(i)\right].$$

Assume that t_0 is the smallest integer for which

$$P\{b^d(t_0)>\psi^0(R^d(t_0))\}=0.$$

Since $b^d(0)=P\{\tau^d<\infty\}>\psi^0(s)=\psi^0(R^d(0))$ we have $t_0>0$. From $P\{b^d(t_0-1)>\psi^0(R^d(t_0-1))\}>0$ we derive $P\{d(t_0-1)>0\}>0$. On the set $\{d(t_0-1)>0\}$ we let

$$\begin{aligned}f(t_0-1)&=d(t_0-1)-1,\\f(t)&=1\text{ if }X_t<c,t\geq t_0,\\f(t)&=d(t)\text{ elsewhere.}\end{aligned}$$

Then $\tau^f \geq \tau^d$, and

$$E\left[\sum_{i=0}^{\infty} v^i f(i)\right] - E\left[\sum_{i=0}^{\infty} v^i d(i)\right]$$
$$\geq \sum_{i=t_0}^{\infty} v^i P\{X < c\}^i = v^{t_0-1}\frac{vP\{X < c\}}{1 - vP\{X < c\}} - v^{t_0-1} > 0.$$

Now we can prove continuity of the function $g : \alpha \to u(s, \alpha)$ for fixed s. Since g is zero on the interval $0 < \alpha \leq \psi^0(s)$ and non decreasing on $0 < \alpha \leq 1$ we need to show that there is no $\alpha_0 \in (\psi^0(s), 1]$ and $\varepsilon > 0$ such that for all $\alpha < \alpha_0$ we have $g(\alpha) < g(\alpha_0) - \varepsilon$. Assume that we can find such values. Then there exists a dividend strategy $d(t)$ satisfying $\psi^d(s) = \alpha_0 > \psi^0(s)$ such that for all $\alpha < \alpha_0$ we have

$$E\left[\sum_{i=0}^{\infty} v^i d(i)\right] > g(\alpha_0) - \varepsilon.$$

Then we can find a dividend strategy $f(t)$ for which

$$E\left[\sum_{i=0}^{\infty} v^i d(i)\right] \leq E\left[\sum_{i=0}^{\infty} v^i f(i)\right]$$

satisfying $P\{b^f(t) > \psi^0(R^f(t))\} > 0,\ t = 0, 1, 2, ...$Choose T sufficiently large such that

$$E\left[\sum_{i=T}^{\infty} v^i f(i)\right] < \varepsilon/2$$

and define the dividend strategy $f_1(t)$ by $f_1(t) = f(t)1_{\{t<T\}}$. Then $\psi^{f_1}(s) < \psi^f(s)$ (this follows from $P\{b^f(T) > \psi^0(R^f(T))\} > 0$) and

$$E\left[\sum_{i=0}^{\infty} v^i f_1(i)\right] \geq \left[\sum_{i=0}^{\infty} v^i f(i)\right] - \varepsilon/2$$

which is contradictory.

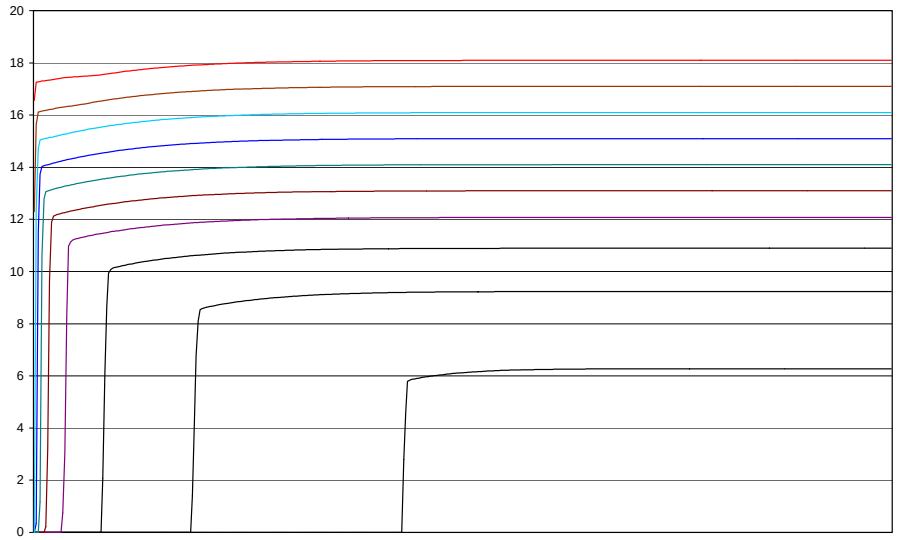
e) Fix s and α. The sup over δ is always attained at some δ^* since it is a sup over a finite number of values. Let $\beta_n(x)$ be a sequence of functions with

$$u(s, \alpha) = \lim_n \{\delta^* + vE[u(s - \delta^* + c - X, \beta_n(X))]\}.$$

There exists a subsequence along which the functions converge pointwise: $\beta_{n_i}(x) \to \beta(x)$. Then by continuity and boundedness of $a \to u(s, a)$ we obtain our assetion

$$u(s, \alpha) = \delta^* + vE[u(s - \delta^* + c - X, \beta(X))].$$

Let $d(t)$ be the strategy defined with the maximizers $\delta(s, \alpha), \beta(s, \alpha; x)$ in (4). Then as in c) we can show that the expected accumulated discounted dividends $d(t)$ are given by $u(s, \alpha)$. Instead of an inequality we use equality since the sup is attained at the maximizers. ■



3 Numerical example

Here we consider the special case of a skip free risk process: $c = 1, P\{X_1 = 0\} = 1 - P\{X_1 = 2\} = 0.7$. Using the iteration (9) we computed the functions $u(s,\alpha), \beta_1(s,\alpha) = \beta(s,\alpha;1)$ (the value for $\beta_2(s,\alpha) = \beta(s,\alpha;2)$ can be derived from the martingale condition), and $\delta(s,\alpha)$. For $\alpha \leq \psi^0(s)$ we have set $\beta(s,\alpha;1) = 1$. From the numerical results we derive the following conjectures:

1. $\delta(s,\alpha) = s - s(\alpha)$.
2. $\beta_2(s,\alpha) = \psi^0(s(\alpha) - 1)$ is independent of s as soon as $\psi^0(s-1) \leq \alpha$.
3. $u_\alpha(s,\alpha) = \infty$ at the point $\alpha(s) = \inf\{a : u(s,a) > 0\}$.

The figure shows the value functions $u(s,\alpha)$ for $s = 0, ..., 10$, computed with a step size of $\Delta = 1/450$ (i.e. $u(s,\alpha)$ is approximated at the points $k\Delta, k = 0, ..., 450$), and the range of s is restricted to $s \leq 20$. On the $x-$axis α runs from 0 to 1. For each value of s a separate curve is shown, the top curves belonging to large values of s.

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