A step forward with Kolmogorov

Pension and life insurance liabilities: analytic/numeric methods

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Agenda

- The classic life insurance setup: reserve and cash flow
- The doubly stochastic setup
  - Simulation or analytic/numerical methods?
  - A generalisation of Kolmogorov's forward differential equation
- Forward transition rates
- Cash flows: stochastic and deterministic payments

Introduction

Consider a pension policy

- paid for by a single or level premium
- certain *guaranteed* benefits (life annuity, disability coverage, ...)

We consider two questions:

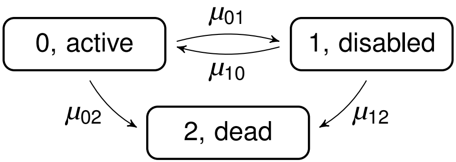
Question 1: What is todays value of the guarantee? Referred to as the *market value* or simply *reserve*

The answer depends on the market interest rate, hence:
Question 2: What is the sensitivity of the *market value* with respect to the interest rate? Easily calculated from the future expected *cash flow*

The classic setup: probabilities

Payments occur in states and upon transitions

- what are the probabilities of the payments occuring?



Probabilities calculated with *Kolmogorov's forward differential eq.*

$$\frac{\mathrm{d}}{\mathrm{d}s} p_{ij}(t,s) = -\sum_{\ell \neq j} p_{ij}(t,s)\mu_{j\ell}(s) + \sum_{\ell \neq j} p_{i\ell}(t,s)\mu_{\ell j}(s)$$

where

$$p_{ij}(t,s) = P(Z(s) = j \mid Z(t) = i).$$

The classic setup: reserve and cash flow

Actual payments at time s

- if in state j: $b_j(s)$
- upon a transition $j \to \ell$: $b_{j\ell}(s)$

Expected payments at time s (the cash flow)

$$a_{t,i}(s) = \sum_{j \in \mathcal{J}} p_{ij}(t,s) \left(b_j(s) + \sum_{\ell \neq j} \mu_{j\ell}(s) b_{j\ell}(s) \right)$$

The reserve: present value of expected payments

$$V_i(t) = \int_t^\infty e^{-\int_t^s r(u)\,\mathrm{d}u} a_{t,i}(s)\,\mathrm{d}s$$

Remark: $V_i(t)$ also solves Thiele's ODE

Stochastic transition rates and payments

- Examples of stochastic transition rates
  - Stochastic mortality
  - Dependent mortality and disability
- Stochastic transition rates: A way to model dependence between policyholders
  - Stochastic mortality affects all policyholders
  - Mass surrender:
    - large customers/brokers moving to another pension provider
- Stochastic transition rates necessary to model (Solvency II) capital requirement
- Stochastic payments: Guaranteed benefits, linked to mortality level

Solvency II rules motivates more advanced modelling.

Doubly stochastic setup

Assume underlying stochastic process,

- $X(t)$ multi-dimensional diffusion process,

$$\mathrm{d}X(t) = \beta(t, X(t))\,\mathrm{d}t + \sigma(t, X(t))\,\mathrm{d}W(t)$$

- Stochastic transition rates: $\mu_{ij}(t, X(t))$
- Given the path of $X(t)$, assume $Z(t)$ is a Markov chain.

Stochastic payments at time t: may depend on $X(t)$:

- if in state j: $b_j(t, X(t))$
- upon a transition $j \to \ell$: $b_{j\ell}(t, X(t))$

What about: stochastic interest rate?

- In this talk: assume *independence* of $X(t)$
- If dependent: Can be handled mathematically, but the cash flow becomes interest rate dependent...

The problem: the cash flow

$$
\begin{aligned}
&a_{t,i_0}(s)\,\mathrm{d}s \\
&= \mathrm{E}_{t,i_0,x_0}\left[\left(b_{Z(s)}(s,X(s))\,\mathrm{d}s + \sum_j b_{Z(s-)j}(s,X(s))\,\mathrm{d}N_{Z(s-)j}(s)\right)\right] \\
&= \sum_i \int p_{i_0 i}(t,x_0,s,x)\left(b_i(s,x) + \sum_{j\neq i}\mu_{ij}(s,x)b_{ij}(s,x)\right)\mathrm{d}x\,\mathrm{d}s
\end{aligned}
$$

Ideas for solving:

1. Simulation of $(Z(t),X(t))$
2. Solve analytically/numerically:
    - The reserve:
        - Thiele's ODE becomes a PDE
    - The cash flow:
        - Kolmogorov's backward ODE becomes a PDE
        - Kolmogorov's forward ODE ...

Kolmogorov's forward partial differential equation

Let $p_{ij}(t_0, x_0, t, x)$ be the joint conditional density of $(Z(t), X(t))$,

$$p_{ij}(t_0, x_0, t, x)\,\mathrm{d}x = P(Z(t) = j, X(t) = \mathrm{d}x | Z(t_0) = i, X(t_0) = x_0)$$

Theorem: Kolmogorov's forward PDE

$$\begin{aligned}
&\frac{\partial}{\partial t} p_{ij}(t_0, x_0, t, x) \\
&= -\sum_{\substack{\ell \in \mathcal{J} \\ \ell \neq j}} p_{ij}(t_0, x_0, t, x)\mu_{j\ell}(t, x) + \sum_{\substack{\ell \in \mathcal{J} \\ \ell \neq j}} p_{i\ell}(t_0, x_0, t, x)\mu_{\ell j}(t, x) \\
&\quad - \sum_{k=1}^{d} \frac{\partial}{\partial x_k} (\beta_k(t, x) p_{ij}(t_0, x_0, t, x)) \\
&\quad + \frac{1}{2} \sum_{k,m=1}^{d} \frac{\partial^2}{\partial x_k \partial x_m} (\Sigma_{km}(t, x) p_{ij}(t_0, x_0, t, x))
\end{aligned}$$

Cash flows in the doubly stochastic setup

Generalisation of the classic setup:

- Stochastic transition rates $\mu_{ij}(t, X(t))$
- Stochastic payments $b_i(t, X(t))$, $b_{ij}(t, X(t))$

Theorem: The cash flow

$$a_{t_0,i,x_0}(t) = \sum_{j \in \mathcal{J}} \int p_{ij}(t_0, x_0, t, x) \Big(b_j(t, x) + \sum_{\ell \neq j} \mu_{j\ell}(t, x) b_{j\ell}(t, x) \Big) \, \mathrm{d}x$$

Compare with the classic setup

$$a_{t_0,i}(t) = \sum_{j \in \mathcal{J}} p_{ij}(t_0, t) \Big(b_j(t) + \sum_{\ell \neq j} \mu_{j\ell}(t) b_{j\ell}(t) \Big)$$

Forward transition rates

Definition: Forward transition rate

The *forward transition rate* $f_{t_0}^{ij}(t)$, measured at time t_0, for the transition rate $\mu_{ij}(t, X(t))$ at time t, is defined as

$$f_{t_0}^{ij}(t) = \lim_{h \searrow 0} \frac{1}{h} P\big(Z(t+h) = j \,|\, Z(t) = i, (Z, X)(t_0)\big)$$

Lemma: Forward transition rate as expectation

$$f_{t_0}^{ij}(t) = \mathrm{E}[\mu_{ij}(t, X(t)) | \, Z(t) = i, (Z, X)(t_0)]$$

In the classic setup: Simplifies to the transition rates

$$\mu_{ij}(t) = \lim_{h \searrow 0} P(Z(t+h) = j \mid Z(t) = i)$$

Forward transition rates

The joint conditional density of $(Z(t), X(t))$:

$$p_{ij}(t_0, x_0, t, x)\,\mathrm{d}x = P(Z(t) = j, X(t) = \mathrm{d}x \,|\, Z(t_0) = i, X(t_0) = x_0)$$

The conditional density of $Z(t)$ only:

$$p_{ij}(t_0, x_0, t) = \int p_{ij}(t_0, x_0, t, x)\,\mathrm{d}x$$

The forward transition rates replace the usual transition rates in the formulae. In particular:

Theorem: Kolmogorov's forward ODE with forw. rates

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} p_{ij}(t_0, x_0, t) \\ &= \sum_{\ell;\ell\neq j} f^{\ell j}_{t_0,i,x_0}(t) p_{i\ell}(t_0, x_0, t) - \sum_{\ell;\ell\neq j} f^{j\ell}_{t_0,i,x_0}(t) p_{ij}(t_0, x_0, t)\end{aligned}$$

Forward transition rates: the mortality rate

- Survival model with stochastic mortality:

$$\begin{tikzcd} \boxed{0, \text{ active}} \arrow[rr, bend left, "\mu_{01}(t, X(t))"] & & \boxed{1, \text{ disabled}} \end{tikzcd}$$

Forward mortality rate $g_{t_0}(t)$, defined by

$$p_{00}(t_0, t) = \mathrm{E}\left[e^{-\int_{t_0}^{t} \mu_{01}(s, X(s))\,\mathrm{d}s} \,\middle|\, X(t_0) \right] = e^{-\int_{t_0}^{t} g_{t_0}(s)\,\mathrm{d}s}$$

The forward transition rate specialises

$$g_{t_0}(t) = f_{t_0}^{01}(t) = \mathrm{E}\left[\mu_{01}(t, X(t)) \middle| Z(t) = 0, (Z, X)(t_0)\right].$$

The forward mortality rate is the expected transition rate, conditional on being alive.

Cash flows in the doubly stochastic setup

Stochastic payments $b_i(t, X(t))$, $b_{ij}(t, X(t))$

$$a_{t_0,i,x_0}(t) = \sum_{j \in \mathcal{J}} \int p_{ij}(t_0, x_0, t, x) \Big(b_j(t, x) + \sum_{\ell \neq j} \mu_{j\ell}(t, x) b_{j\ell}(t, x) \Big) \, \mathrm{d}x$$

Deterministic payments $b_i(t)$, $b_{ij}(t)$

$$a_{t_0,i,x_0}(t) = \sum_{j \in \mathcal{J}} p_{ij}(t_0, x_0, t) \Big(b_j(t) + \sum_{\ell \neq j} f_{t_0}^{j\ell}(t) b_{j\ell}(t) \Big)$$

Compare with the cash flow in the classic setup

$$a_{t_0,i}(t) = \sum_{j \in \mathcal{J}} p_{ij}(t_0, t) \Big(b_j(t) + \sum_{\ell \neq j} \mu_{j\ell}(t) b_{j\ell}(t) \Big)$$

Thank you!

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