The Φ-Martingale

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Motivation

- Prices = conditional expectations = martingales
- Continuous martingales = key tool in mathematical finance
- Typically, they belong to $\mathbb{R}$ (W_t) or $\mathbb{R}^+$ ($\mathcal{E}_t$)

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- Prices = conditional expectations = martingales
- Continuous martingales = key tool in mathematical finance
- Typically, they belong to $\mathbb{R}$ (W_t) or $\mathbb{R}^+$ ($\mathcal{E}_t$)
- However, they can be restricted to compact sets $C(t)$
- Examples for $C(t) = I = [0, 1]$ if positive rates
  1. Discounted price of risk-free ZCB, $B_t := \mathrm{e}^{-\int_{s=0}^{t} r(s)ds}$

$$M_{t,T} := B_t P(t, T) = \mathbb{E}_t[B_T P(T, T)] = \mathbb{E}_t[B_T] \in I \text{ a.s.}$$

  2. Discounted price of contingent ZCB

$$M_{t,T} := \mathbb{E}_t[B_T \, \mathbb{1}_{\{X_T \in A\}}] \in I \text{ a.s.}$$

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Credit risk modeling : τ a default time

- Suppose $\mathbb{F}$ is the market filtration (excl. default)
- Let $\mathbb{G}$ the progressive enlargement of $\mathbb{F}$ with τ
- Then, τ is a $\mathbb{G}$-stopping time but not a $\mathbb{F}$-stopping time

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Let us note $\mathbb{E}_t[.] := \mathbb{E}[.|\mathcal{F}_t]$

- Define $S_{t,T} := \mathbb{E}_t[\mathbb{1}_{\{\tau>T\}}]$ a $(\mathbb{Q},\mathbb{F})$-martingale $\in I$ a.s.
- Set $S_t := S_{t,t}$ leading to $S_{t,T} := \mathbb{E}_t[\mathbb{E}_T[\mathbb{1}_{\{\tau>T\}}]] = \mathbb{E}_t[S_T]$

$$\mathbb{E}[X\,\mathbb{1}_{\{\tau>T\}}\,|\mathcal{G}_t] = \frac{\mathbb{1}_{\{\tau>t\}}}{S_t}\,\mathbb{E}_t[XS_T]$$

$$\mathbb{E}[X_\tau\,\mathbb{1}_{\{t<\tau<T\}}\,|\mathcal{G}_t] = -\frac{\mathbb{1}_{\{\tau>t\}}}{S_t}\,\mathbb{E}_t\left[\int_t^T X_u dS_u\right]$$

Modeling of S_t

- Let $T \downarrow t$: $S_t = S_{t,t}$ is the Azéma supermartingale
- General Doob-Meyer decomposition

$$S_t = D_t + M_t$$

- D_t $\mathcal{F}_t$-predictable decreasing, $M_t \in (\mathbb{Q}, \mathbb{F})$-martingale

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- In Cox setup: $D_t = \mathrm{e}^{-\Lambda_t}$ and $M_t \equiv 0$
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- In practice, not easy to find $\lambda_t \geqslant 0$ s.t. $\mathbb{E}[S_t] = S_0(t)$
- Idea: model $S_{t,T}$ s.t. $S_{0,T} = S_0(T)$, since then:

$$\mathbb{E}[S_T] = \mathbb{E}[S_{T,T}] = S_{0,T} = S_0(T)$$

- Cesari: $S_{t,T}$ Gaussian easy but $\mathbb{Q}\{S_{t,T} \notin I\} > 0$
- Couldn't we find a better solution ?

Outline

- Local martingale, martingale and solutions to SDE
- The Φ-martingale : a $[0, 1]$ process
- Drawing sample paths
- Asymptotic distribution
- Example: Survival probability
- Conclusion and references

The questions we want to address

- Aim: find a tractable martingale process Y_t evolving in I
- Idea: study "smooth" SDEs of the form

$$dY_t = \sigma(t, Y_t)dW_t \quad (E)$$

where $\sigma(t, x) = 0$ when $x \in \partial I = \{0, 1\}$ and $Y_0 \in I$

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- Facts:
  - One can show (via continuity argument) that $Y_t \in I \; \forall t \geqslant 0$
  - The solution Y_t (if it exists) is a *local* martingale
- However, a lot of open questions
  - Do such types of SDEs admit a solution ?
  - If so, is the analytical form of Y_t available ?
  - If solution exists, is it a (genuine) martingale ?
  - If merely a *local* martingale, who cares ?

Local Martingales *vs* (Global) Martingales

- Local martingales are not necessarily martingales
- Cox & Hobson: we really need to have martingales

  In a market with a bubble (price process being a local martingale, but not a martingale), many standard results from the folklore become false. Put-call parity fails (. . .)

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  In a market with a bubble (price process being a local martingale, but not a martingale), many standard results from the folklore become false. Put-call parity fails (. . .)
- If it exists, the solution to the *driftless* SDE

$$dY_t = \sigma(t, Y_t)dW_t$$

  is a martingale if $Y_0 \in \mathbb{R}$ and

$$\mathbb{E}\left[\int_{s=0}^{t} \sigma^2(s, Ys)ds\right] < \infty \quad (M)$$

- Problem: condition (M) hard to check and "easily" broken
- George Lowther's example: $\mathbb{E}_t[Y_s] < Y_t$ if $\sigma(t, x) = x^2$

Bounded martingales

- Do you know the solution of a SDE of the form (E) ?
- Can we even find $\sigma(t,x)$ such that a solution exists ?
- Good news: if the solution exists, it is a genuine martingale
- Bounded local martingales are martingales

$\sigma(t,x)$	sol $\exists$?	martingale ?
$kx(1-x)$	✓	✓
$kx\ln(x)$	?	✓ if $\exists$

- $|x(1-x)|$ is Lipschitz and bounded on I ($Y_t \in I$) but, $Y_t =$? How to simulate: $Y_t \in I$?
- Idea: use $Y_t = F(X_t)$ where $X_t \in \mathbb{R}$

Construction of $[0, 1]$-martingale via Ito-Doeblin

- Let $F(x) : \mathbb{R} \to I$ a $\mathcal{C}^2$ bijection with $F'(x) = f(x)$, $F''(x) = f'(x)$, $|f(x)| < \infty$ and $\psi(x) \doteq -\frac{f'(x)}{f(x)}$
- Assume $X_t \in \mathbb{R}$ solves the *well-behaved* SDE

$$dX_t = \mu(t, X_t)dt + \eta(t, X_t)dW_t$$

- Then $Y_t = F(X_t) \in I$ is a martingale iff

$$\mu(t, x) = \frac{\eta^2(t, x)}{2}\psi(x)$$

- Moreover, Y_t satisfies (E) with $Y_0 = F^{-1}(X_0)$ and

$$\sigma(t, x) = \eta(t, F^{-1}(x))f \circ F^{-1}(x)$$

- Ex: Let $F(x) = (1 + \tanh(x))/2$. Then
  - $Y_t = F(X_t) \in I$ is a martingale iff $\mu(t, x) = \eta(t, x)^2 \tanh(x)$
  - Since ψ is Lipschitz, X_t exists if η regular enough

A corollary useful for simulation

- Suppose we want to find the solution Y_t the following SDE

$$dY_t = g(t)h(Y_t)dW_t \quad (EY)$$

- Assume $Y_0 \in (0,1)$, $|g(t)| < \infty$, $h(x) \in \mathcal{C}^2$, $|h(x)| > 0$ on $(0,1)$ and vanishes on ∂I
- Let $F(x)$ be a smooth solution to ODE $F'(x) = h(F(x))$
- Then, $Y_t = F(X_t)$ where X_t is the solution to

$$dX_t = \frac{g^2(t)}{2}\psi(x)dt + g(t)dW_t \quad (EX)$$

- Ex: $g(t) = \sigma, h(x) = x(1-x) \Rightarrow F(x) = ce^x/(1+ce^x)$
  - $Y_t = F(X_t)$ where X_t is the unique solution to (EX) with $\psi(x) = \tanh(x/2)$ (Lipschitz)
  - Useful for drawing $Y_t \in I$, but expression of X_t unknown

A simple example with known solution

- Idea: set $\eta(t,x) = \sigma$ & find $F : \psi$ is affine (X_t Gaussian)
- Let ϕ and Φ denote the standard Normal pdf and cdf
- Taking $F = \Phi$ leads to $\psi(x) = x$ (X_t Vasicek process)

$$\begin{aligned} dX_t &= \sigma^2/2 X_t + \sigma dW_t \\ X_t &= X_0 \, \mathrm{e}^{\frac{\sigma^2}{2}t} + \sigma \int_{u=0}^{t} \mathrm{e}^{\frac{\sigma^2}{2}(t-u)} \, dW_u \\ &\sim \mathcal{N}\left(X_0 \, \mathrm{e}^{\frac{\sigma^2}{2}t}, \sqrt{\mathrm{e}^{\sigma^2 t} - 1} \right) \end{aligned}$$

- This leads to the SDE (E) for Y_t with diffusion coefficient

$$\sigma(t,x) = \sigma\phi \circ \Phi^{-1}(x), \;\; Y_0 = \Phi(X_0)$$

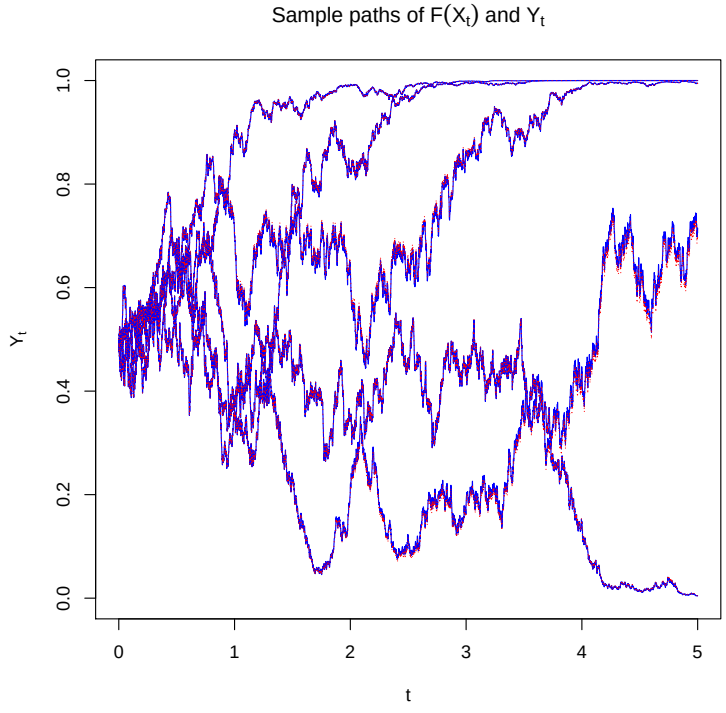
- *If it exists*, the solution $Y_t \in I$ is a martingale
- $\sigma(t,x)$ not Lipschitz but solution $Y_t = \Phi(X_t)$ is unique

A simple example: sample paths

Let $Y_0 \in (0,1)$, $X_0 = \Phi^{-1}(Y_0)$ and

$$dY_t = \sigma\phi \circ \Phi^{-1}(Y_t)dW_t \quad \Rightarrow \quad Y_{(n+1)\delta} = Y_{n\delta} + \sigma\phi \circ \Phi^{-1}(Y_{n\delta})\sqrt{\delta}\epsilon_n$$

$$dX_t = \sigma^2/2X_t + \sigma dW_t \quad \Rightarrow \quad X_{(n+1)\delta} = (1 + \sigma^2/2)X_{n\delta} + \sigma\sqrt{\delta}\epsilon_n$$



Solution X_t is known: we can draw X_t and get $\Phi(X_t) = Y_t \in I$

Asymptotic distribution ($F = \Phi$, $\sigma = 0.5$)

The p-quantile of Y_t is $q(t,p) : \mathbb{Q}\{Y_t \leqslant q(t,p)\} = p$:

$$\Phi\left(e^{\sigma^2/2t}\left(\frac{\sqrt{e^{\sigma^2 t}-1}}{e^{\sigma^2/2t}}\Phi(p)+\Phi(Y_0)\right)\right) \overset{t\to\infty}{\to} \frac{\mathbb{1}_{\{p\geqslant 1-Y_0\}}+\mathbb{1}_{\{p>1-Y_0\}}}{2}$$

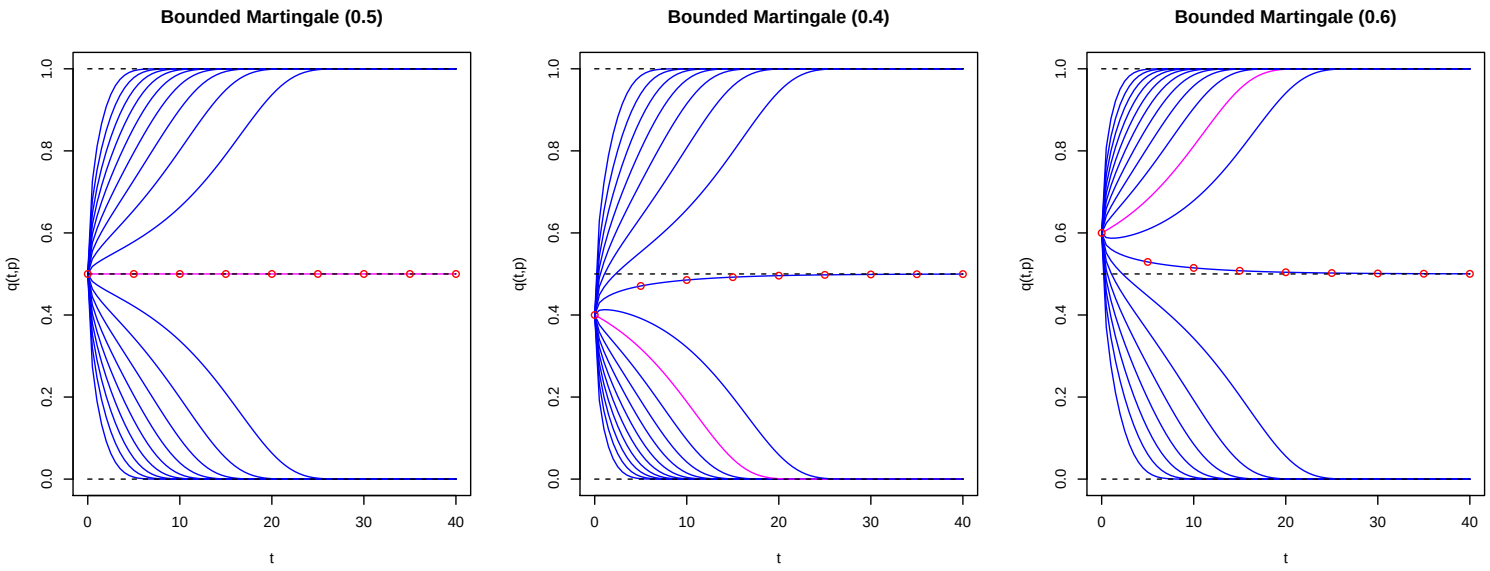


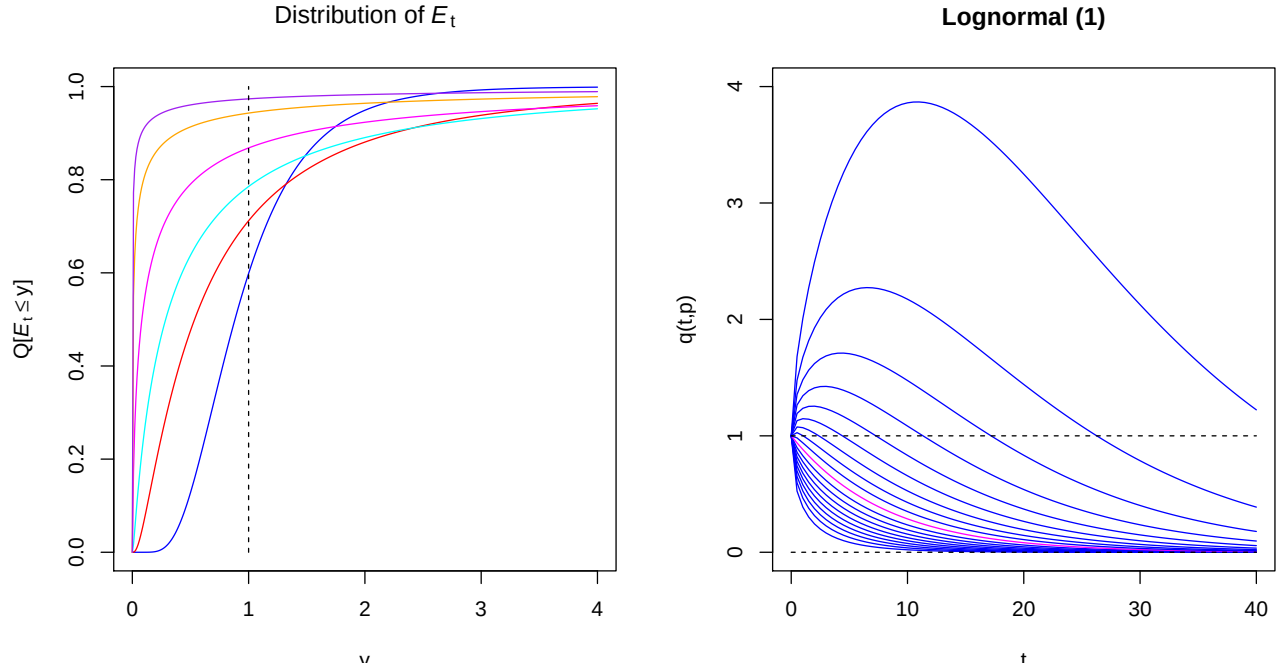
Figure: Quantiles (blue), median (magenta) and $q(t, 1-Y_0)$ (red dots)

One can prove that $Y_t \overset{\mathbb{Q}}{\to} Bern(Y_0)$ *as* $t \to \infty$

Asymptotic behavior of the exp. martingale $\mathcal{E}_t$

- $\mathcal{E}_t = \mathrm{e}^{-\sigma^2 t/2 + \sigma W_t}$: $\mathbb{E}[\mathcal{E}_t] = 1$, $\mathrm{var}[\mathcal{E}_t] = \mathrm{e}^{\sigma^2 t} - 1 \overset{t\to\infty}{\to} \infty$
- $\mathcal{E}_t$ tends to fall: $\mathbb{Q}\{\mathcal{E}_t \leqslant \mathcal{E}_0\} > \mathbb{Q}\{\mathcal{E}_t \leqslant e^{-\sigma^2 t/2}\} = 1/2$
- The p-quantile of $\mathcal{E}_t$ is given by

$$q(t,p) = \exp\{\sigma\sqrt{t}\Phi^{-1}(p) - \sigma^2/2t\} \overset{t\to\infty}{\to} 0$$



- The longer tail is compensated by a higher peak at $x = 0$
- So, the whole distribution collapses to $x = 0$

Autonomous Gaussian martingales

- Definition: The martingale Y_t is said autonomous Gaussian iff $Y_t = F(X_t)$ where
  1. X_t is a Gaussian process
  2. $F(x)$ is a bijection
  3. $dY_t = g(t)h(Y_t)dW_t$ for some smooth g, h
- Theorem: The only autonomous Gaussian martingales are, up to shifting and rescaling,
  1. Brownian motions
  2. Geometric Brownian motions
  3. Φ-martingales
- Interestingly, each type of process Y_t corresponds to a specific range, respectively
  1. $\mathbb{R}$ (or 0 for the trivial rescaling $0W_t$)
  2. Semi-closed interval $[a, \infty)$ or $(-\infty, a]$
  3. Closed interval $[a, b]$

Example: Survival probability - $S_{t,T}$

- We need to find a martingale $S_{t,T} \in I$ s.t. $S_{0,T} = S_0(T)$
- Set $S_{t,T} = \Phi(X_{t,T}) \in I$:

$$X_{t,T} = \Phi^{-1}(S_0(T))\, e^{\mu t} + \sigma Z_t \ , \quad Z_t = \int_{s=0}^{t} e^{\mu(t-s)}\, dW_s$$

- $S_{t,T} \in I$, $S_{0,T} = S_0(T)$ and local martingale if $\mu = \sigma^2/2$
- $S_{t,T}$ bounded local martingale $\Rightarrow$ martingale
- Density $f_{X_{t,T}}$ of $X_{t,T}$ known (Gaussian), so is that of Y_t:

$$f_{S_{t,T}}(y) = \frac{1}{\phi \circ \Phi^{-1}(y)} f_{X_{t,T}}(\Phi^{-1}(y))$$

- Both $S_{t,T}$ in $[0,1]$ and T-decreasing for all $0 \leqslant t \leqslant T$

Dynamics of S_t ($t > 0,\ \mu = \sigma^2/2$)

- Set $S_t = \Phi(X_t)$ where $X_t := X_{t,t}$. Then

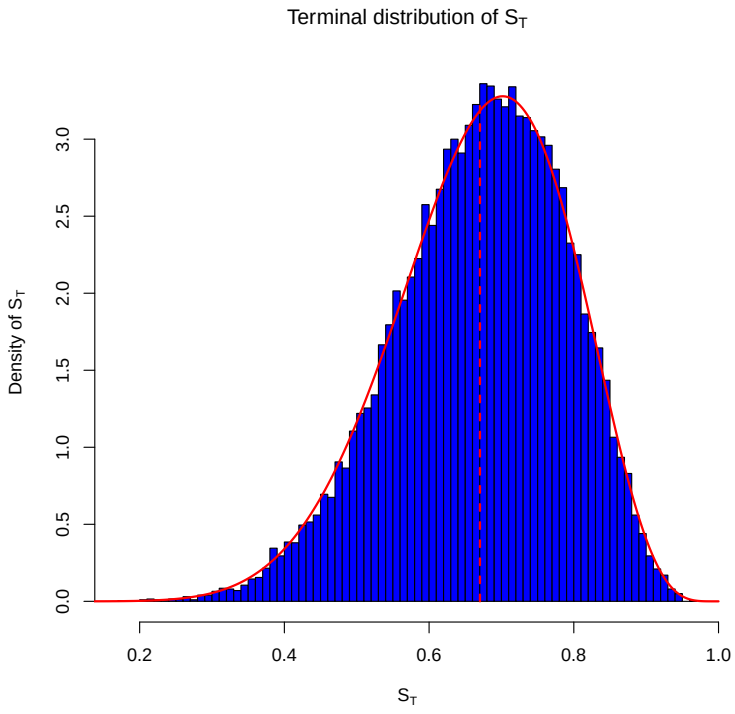
$$dS_t = \underbrace{\zeta_t dS_0(t)}_{dD_t} + \underbrace{\sigma\phi(\Phi^{-1}(S_t))dW_t}_{dM_t}, \quad \zeta_t = \frac{\phi(\Phi^{-1}(S_t))\,\mathrm{e}^{\mu t}}{\phi(\Phi^{-1}(S_0(t)))}$$

  where $\zeta_0 = 1$ and $\mathbb{E}[\zeta_t] = 1$

- Therefore, with $\tau > 0$ and $\mathcal{G}_0 = \mathcal{F}_0$ we have

$$\begin{aligned}
\mathbb{E}\left[V_\tau\, 1\!\!1_{\{\tau \leqslant T\}}\right] &= -\mathbb{E}\left[\int_0^T V_t dS_t\right] \\
&= -\mathbb{E}\left[\int_0^T V_t dD_t + \int_0^T V_t dM_t\right] \\
&= -\int_0^T \mathbb{E}\left[\zeta_t V_t\right] dS_0(t)
\end{aligned}$$

Azéma supermartingale $S_t = \Phi(X_t)$



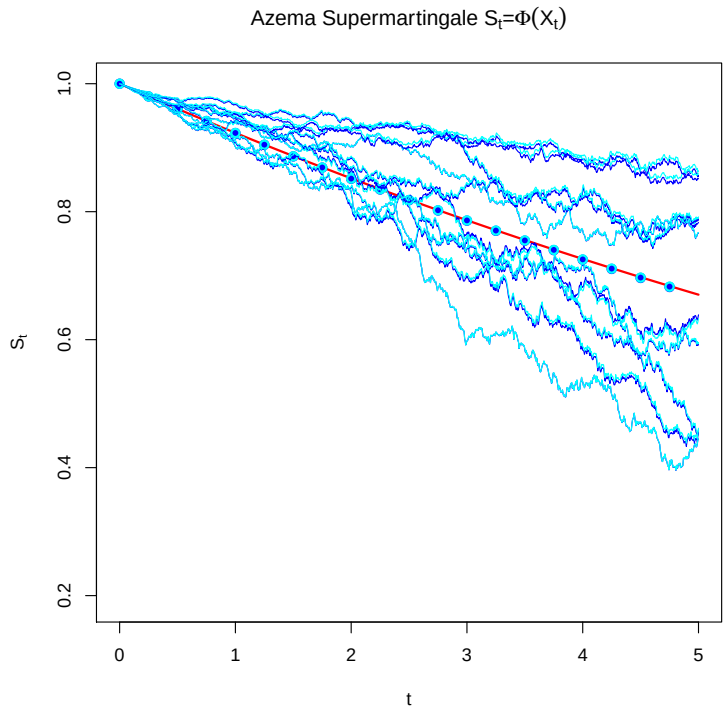


Table: Left: Density of S_T. Histogram (10k) and theoretical density (red); Right: sample paths of $S_t = \Phi(X_t)$ (cyan) and using Euler discretization of S_t SDE (dark blue). In both cases, $\hat{\mathbb{E}}[S_t] = S_0(t)$ (red). Params: $\sigma = 0.15,\ \mu = \sigma^2/2,\ S_0(t) = \mathrm{e}^{-ht},\ h = 8\%,\ T = 5Y$.

Conclusion

- We were looking for a way to obtain martingales in $[0, 1]$
- We focused on SDEs without drift (local martingales)
- If solution $\exists$, it will be a genuine martingale
- Typically: solution not guaranteed or unknown

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- We focused on SDEs without drift (local martingales)
- If solution $\exists$, it will be a genuine martingale
- Typically: solution not guaranteed or unknown
- Idea map a specific Gaussian process through Φ
- Analytical solution and distribution available
- Further work includes analysis of
  1. calibration flexibility
  2. usefulness in hybrid credit apps (impact of so-called hypothesis (H) being violated)

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