Derivatives Pricing in Incomplete Markets

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Presentation Structure

- Introduction
- Approximation of Incomplete Markets
- Numerical illustration

Introduction

- Reasons why markets are incomplete
- Insufficiency of assets in the market relative to the class of risks that one wish to hedge

  -jumps of volatility;

  -variables that are not derived from market prices
- Market frictions

  -transaction costs

  -portfolio constrains

  -illiquidity

Introduction

- Jeremy Staum, Incomplete Markets,
  Handbooks in OR & MS, Vol.15, 2008
  excellent survey in methodologies of pricing, relationships between them (128 references)
- Harry H. Panjer, editor: Boyle P.P., et al. Financial Economics: With applications to Investments, Insurance and Pensions, 1998.
- T. Bjork, Arbitrage theory in continuous time (2004).
- D.Duffie, Dynamic Asset Pricing Theory (2001)
- Shiriaev A.N. Osnovy finansovoy stohasticheskoy matematiki, tom 2. Moskow, Fazis,1998, 544p. (in Russian).
- Shiriaev V.I. Financial mathematics, option pricing, probabilistic & guaranteed approaches ,2007.(in Russian) .
- A.V. Melnikov,N.V. et al. Mathematical methods of financial analysis, M, Ankip,2006 (in ussian)
- G.A. Medvedev - in Belarus

Introduction

- Main approaches to derivative pricing in incomplete markets

Asset prices are modeled under

- physical probability measure
- risk-neutral probability
- upper and lower bounds for price of derivative

Introduction
Single –period model

Suppose we have a derivative security whose payoff at time 1 is a function of $S_1(\omega_j)$, that is

$$V_j = f(S_1(\omega_j)), \qquad j = 1,2.$$

We can construct a portfolio of these two securities that has the same value as the derivative security in each state at time 1. Assume the portfolio consists of θ_1 units of B_1 and θ_2 units of S_1. We must have

$$V_1 = \theta_1(1+r)B_0 + \theta_2 S_1(\omega_1),$$
$$V_2 = \theta_1(1+r)B_0 + \theta_2 S_1(\omega_2).$$

A unique solution of this system of two linear equations exists because of the complete market assumption and we have

$$\theta_1 = \frac{V_1 S_1(\omega_2) - V_2 S_1(\omega_1)}{(1+r)B_0 S_1(\omega_2) - (1+r)B_0 S_1(\omega_1)}, \qquad (1)$$

$$\theta_2 = \frac{V_2(1+r)B_0 - V_1(1+r)B_0}{(1+r)B_0 S_1(\omega_2) - (1+r)B_0 S_1(\omega_1)} = \frac{V_2 - V_1}{S_1(\omega_2) - S_1(\omega_1)}. \qquad (2)$$

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If there is no an arbitrage, then the time-0 value of the portfolio must be equal to the time -0 price of the derivative. Hence the time-0 price value is

$$V_0 = \theta_1 B_0 + \theta_2 S_0, \qquad (3)$$

where θ_1 and θ_2 are given by (1) and (2). It is convenient to simplify the notations.

Let $S_1(\omega_1) = S_0(1+d)$, $S_1(\omega_2) = S_0(1+u)$. If there is no an arbitrage, then

$u > r > d$, and the equation (3) takes the form

$$V_0 = \frac{f(S_0(1+d))(u-r)}{(1+r)(u-d)} + \frac{f(S_0(1+u))(r-d)}{(1+r)(u-d)}).$$

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Hence

$$V_0 = \frac{1}{1+r} f(S_0(1+d)) \cdot q_1 + \frac{1}{1+r} f(S_0(1+u)) \cdot q_2 \qquad (4)$$

where

$$q_1 = \frac{u-r}{u-d}, \quad q_2 = \frac{r-d}{u-d}.$$

are risk neutral probabilities. Denoting corresponding probability measure

Q, we can rewrite (4) in the form

$$V_0 = E^Q\left[\frac{f(S_1)}{1+r}\right].$$

Approximation of Incomplete Market

From the previous part we can conclude that if the stock price has only two values we can construct the replicating portfolio (θ_1, θ_2), and the derivative price is given by (1)-(3). In [4] Takahashi analyzed the situation where the stock price can take 3 values at time 1 and presented an approach of choosing the martingale measure which is the combination of the method of least squares and the embedded complete markets.

In this part we consider the situation where the stock price at time 1 can take more than two values. The equation (4) could not be applied because a risk neutral measure Q is not unique. In this situation we introduce a fictive market (B, S^*) which is complete, and in some sense is close to our market. In this fictive market the stock price takes only two values. To introduce the fictive market we need a measure of closeness. It is possible to use an expectation of some function $R(y_1, y_2.)$ of $f(S)$ and the payoff of the replicating portfolio in (B, S^*) market.

Approximation of Incomplete Market

More formally, let $S_1 = S_0(1+\rho)$, $S_1^* = S_0(1+\rho^*)$, where ρ and ρ^* are random variables. We assume, that the random variable ρ^* values are u and *d*, $\Pr(\rho^* = u) = 1 - \Pr(\rho^* = d) = p$. The time-1 prices of the securities S_1 and S_1^* are completely determined by $S_{0,}$ ρ , and ρ^*. We can choose the distribution parameters of ρ^* to make the fictive market closer to the (*B, S*) - market. So we have the problem:

$$ER(f(S_1), \theta_1 B_0(1+r) + \theta_2 S_0(1+\rho)) \to \min_{u,d,p} . \qquad (5)$$

In the problem (5) the resulting portfolio depends on u and d . In practice the problem could be solved numerically by using empirical distributions.

Approximation of Incomplete Market

Further we assume that $R(y_1, y_2) = (y_1 - y_2)^2$.

We will look for the random variable ρ^* in the form

$$\rho^* = \begin{cases} u, & if \quad \rho \geq c, \\ d, & if \quad \rho < c. \end{cases} \qquad (6)$$

Thus the problem (5) in our case is to find such ρ^* of the type (6), that minimizes

$$E\left[f(S_0(1+\rho) - f(S_0(1+\rho^*) - \frac{f(S_0(1+u)) - f(S_0(1+d))}{u-d}(\rho - \rho^*) \right]^2 . \qquad (7)$$

Numerical illustration

Having the observations of the $(B_0^n, S_0^n, B_1^n S_1^n)$ in the discrete time moments $n = 1,2,\ldots,N$ and assuming that $\rho_n = \dfrac{S_1^n - S_1^{n-1}}{S_1^{n-1}}$ are independent identically distributed random variables that have the same distribution as ρ we can minimize empirical estimator of (7) numerically.

The problem to be solved is

$$\frac{1}{N}\sum_{n=1}^{N}\left[f(S_0^n(1+\rho_n)) - f(S_0^n(1+\rho_n^*)) - \frac{f(S_0^n(1+u)) - f(S_0^n(1+d))}{u-d}(\rho_n - \rho_n^*)\right]^2 \to$$

$$\to \min_{u,d,c} \qquad (8)$$

where

$$\rho_n^* = \begin{cases} u, & if \quad \rho_n \ge c, \\ d, & if \quad \rho_n < c. \end{cases}$$

Numerical illustration

Here we assume that a derivative security is a 1- period European call option. Then

$$f(S_1) = \max(S_1 - K, 0)\,,$$

where K -is the strike price of the call. We have performed the calculations for two choices of K. For the first one $K^1 = K_n = S_0^n(1+\overline{\rho}), \quad n = 1,2,\ \ldots,\ N$,

$\overline{\rho} = \frac{1}{N}\sum_{n=1}^{N}\rho_n$, for the second : $K^2 = \frac{1}{N}\sum_{n=1}^{N} S_0^n$.

From (1)-(2) for the first choice of K the replicating portfolio approximation is given by the equalities

$$\theta_1^n = \frac{\max(S_0^n(1+d^*) - S_0^n(1+\overline{\rho}),0)(1+u^*) - \max(S_0^n(1+u^*) - S_0^n(1+\overline{\rho}),0)(1+d^*)}{u^* - d^*}$$

$$\theta_2^n = \frac{\max(u^* - \overline{\rho},0) - \max(d^* - \overline{\rho},0)}{u^* - d^*}.$$

Numerical illustration

For the second choice of K

$$\theta_1^n = \frac{\max(S_0^n(1+d^*) - K^2, 0)(1+u^*) - \max(S_0^n(1+u^*) - K^2, 0)(1+d^*)}{u^* - d^*},$$

$$\theta_2^n = \frac{\max(S_0^n(1+u^*) - K^2, 0) - \max(S_0^n(1+d^*) - K^2, 0)}{u^* - d^*}.$$

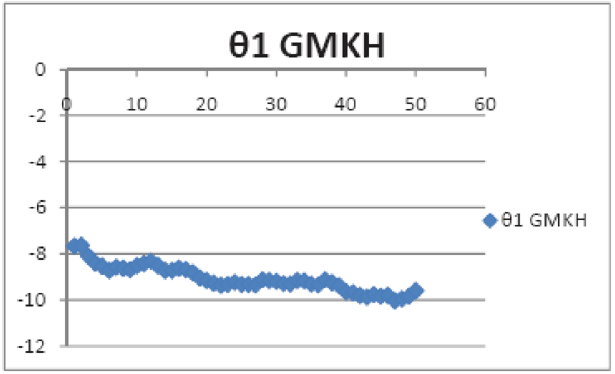
For the numerical illustration we consider the daily data of 3 Russian stocks SBER, GMKH, EERS since 01.04. 2003. We use N=50, and a derivative security is 1-month European call option.

Numerical illustration

Table1

Stock	d^*	u^*	c^*	q_2	Error	Coefficient of variation	θ_2^n
SBER	-0,004843	0,104167	0,02457	0,044424	2.447484	0,478456	0.577214
GMKH	-0.032258	0.078838	0.071584	0.290361	0.484335	0.758302	0.390904
EERS	-0.027388	0.145867	0.042231	0.158087	0.003529	0.751422	0.555813

Figure 1



Numerical illustration

Figure 2

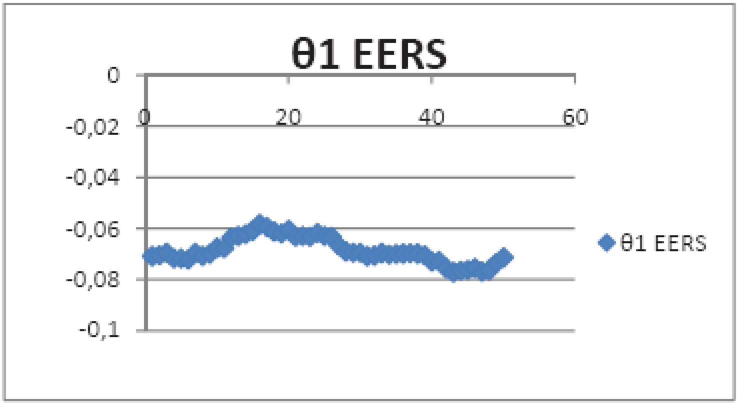
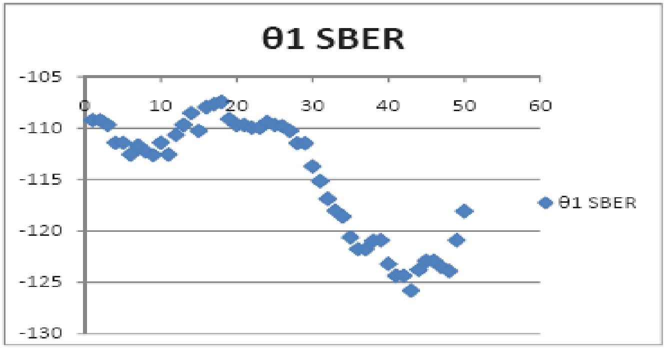


Figure 3



Numerical illustration

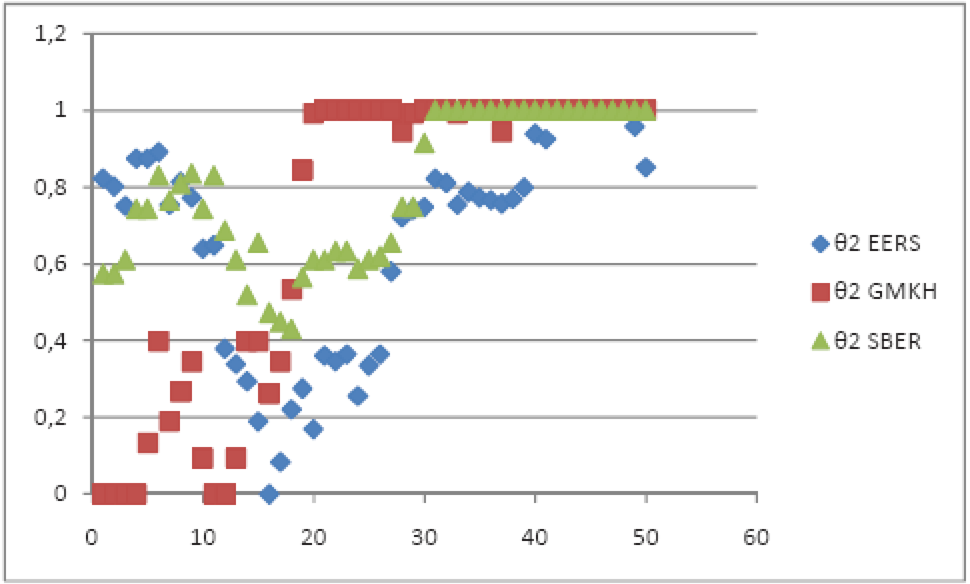
In Table2 the parameters of $(B, S^*)-$ market for $K = K^2$ are given

Table 2

Stock	d^*	u^*	c^*	q_2	Error	Coefficient of variation
SBER	-0,000927	0,120419	0,006835	0,007641	1.34778	0,147891
GMKH	-0,0082988	0,075051	0,071584	0,099565	0,226183	0,29262
EERS	-0.07635	0.17636	-0.0735	0.302113	0.003561	0.533157

Numerical illustration

Figure 4



Numerical illustration

Figures 5-7 illustrate the dependence θ_1^n on n.

Figure 5

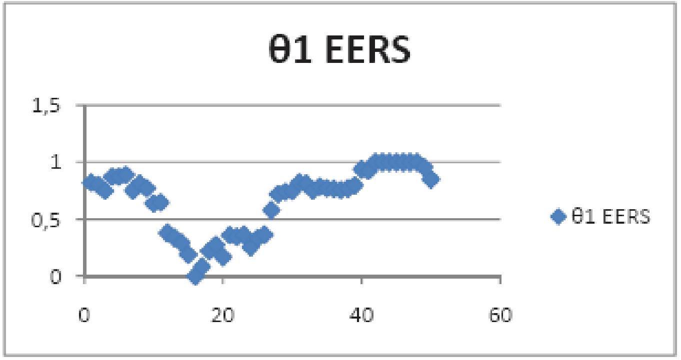


Figure 6

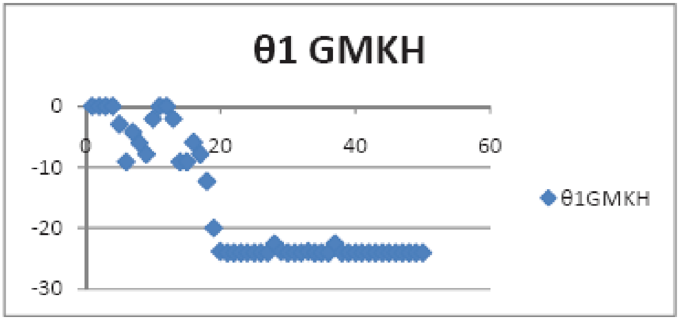
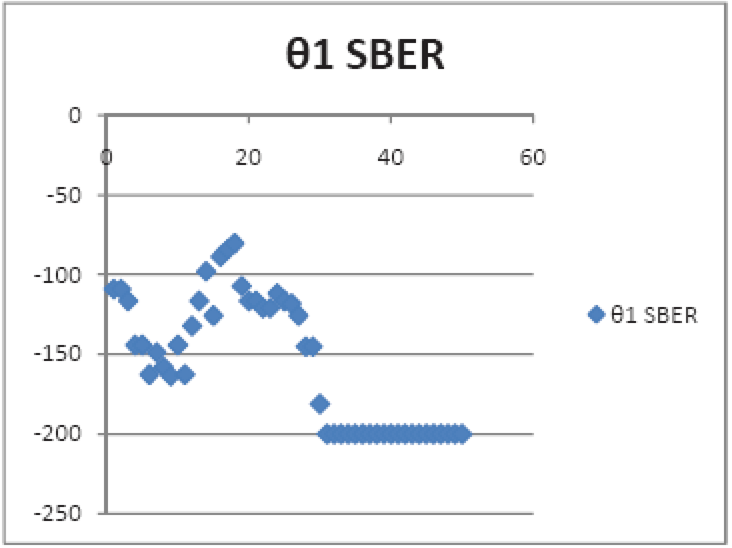


Figure 7



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