DERIVATIVES PRICING IN INCOMPLETE MARKETS

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In the paper the one-period (*B*, *S*) - market model with two securities is considered. In the introduction a replicating portfolio for the derivative security is obtained when the securities are primitive. In the second part we consider the situation where the stock price could have more than two values. We build an approximation of such a market using a fictive (B, S^*)-market where the stock price could take only two values. In the third part the numerical illustration of the approximation is given.

Key words: incomplete market, derivative pricing, call option

1. Introduction

We consider the one- period market model where there are two assets and two outcomes. Their time-0 prices are B_0, and S_0. At time 1 their prices are $B_1(\omega_j) = (1+r)B_0$, and $S_1(\omega_j),\ r \geq 0,\ j = 1,2$. We note that the first security is riskless and the second security is risky. Such a model is a special case of (B, S) -market (see, for example, [1,3]). We assume that all prices are strictly positive and all securities are infinitely divisible. There are no transaction costs. To ensure a complete market, the returns on these securities must be linearly independent. This implies that the payoff matrix

$$\begin{bmatrix} B_1(\omega_1) & S_1(\omega_1) \\ B_1(\omega_2) & S_1(\omega_2) \end{bmatrix}$$

is nonsingular so that its determinant is nonzero.

Suppose we have a derivative security whose payoff at time 1 is a function of $S_1(\omega_j)$, that is

$$V_j = f(S_1(\omega_j)), \qquad j = 1,2.$$

We can construct a portfolio of these two securities that has the same value as the derivative security in each state at time 1. Assume the portfolio consists of θ_1 units of B_1 and θ_2 units of S_1. We must have

$$V_1 = \theta_1(1+r)B_0 + \theta_2 S_1(\omega_1),$$
$$V_2 = \theta_1(1+r)B_0 + \theta_2 S_1(\omega_2).$$

A unique solution of this system of two linear equations exists because of the complete market assumption and we have

$$\theta_1 = \frac{V_1 S_1(\omega_2) - V_2 S_1(\omega_1)}{(1+r)B_0 S_1(\omega_2) - (1+r)B_0 S_1(\omega_1)}, \tag{1}$$

$$\theta_2 = \frac{V_2(1+r)B_0 - V_1(1+r)B_0}{(1+r)B_0 S_1(\omega_2) - (1+r)B_0 S_1(\omega_1)} = \frac{V_2 - V_1}{S_1(\omega_2) - S_1(\omega_1)}. \tag{2}$$

If there is no an arbitrage, then the time-0 value of the portfolio must be equal to the time -0 price of the derivative. Hence the time-0 price value is

$$V_0 = \theta_1 B_0 + \theta_2 S_0, \tag{3}$$

where θ_1 and θ_2 are given by (1) and (2). It is convenient to simplify the notations.

Let $S_1(\omega_1) = S_0(1+d)$, $S_1(\omega_2) = S_0(1+u)$. If there is no an arbitrage, then

$u > r > d$, and the equation (3) takes the form

$$V_0 = \frac{f(S_0(1+d))(u-r)}{(1+r)(u-d)} + \frac{f(S_0(1+u))(r-d)}{(1+r)(u-d)}).$$

Hence

$$V_0 = \frac{1}{1+r} f(S_0(1+d)) \cdot q_1 + \frac{1}{1+r} f(S_0(1+u)) \cdot q_2 \tag{4}$$

where

$$q_1 = \frac{u-r}{u-d}, \quad q_2 = \frac{r-d}{u-d}.$$

are risk neutral probabilities. Denoting corresponding probability measure

Q, we can rewrite (4) in the form

$$V_0 = E^Q\left[\frac{f(S_1)}{1+r}\right].$$

This equation is well known (see, for example [2]). Further without loss of the generality we can assume $B_0 = B_1 = 1$ (see [1, 3]), and $r = 0$.

2. Approximation of incomplete markets.

From the previous part we can conclude that if the stock price has only two values we can construct the replicating portfolio (θ_1, θ_2), and the derivative price is given by (1)-(3). In [4] Takahashi analyzed the situation where the stock price can take 3 values at time 1 and presented an approach of choosing the martingale measure which is the combination of the method of least squares and the embedded complete markets.

In this part we consider the situation where the stock price at time 1 can take more than two values. The equation (4) could not be applied because a risk neutral measure Q is not unique. In this situation we introduce a fictive market (B, S^*) which is complete, and in some sense is close to our market. In this fictive market the stock price takes only two values. To introduce the fictive market we need a measure of closeness. It is possible to use an expectation of some function $R(y_1, y_2.)$ of $f(S)$ and the payoff of the replicating portfolio in (B, S^*) market.

More formally, let $S_1 = S_0(1+\rho)$, $S_1^* = S_0(1+\rho^*)$, where ρ and ρ^* are random variables. We assume, that the random variable ρ^* values are u and d, $\Pr(\rho^* = u) = 1 - \Pr(\rho^* = d) = p$. The time-1 prices of the securities S_1 and S_1^* are completely determined by $S_{0,}$ ρ, and ρ^*. We can choose the distribution parameters of ρ^* to make the fictive market closer to the (B, S) - market. So we have the problem:

$$ER(f(S_1), \theta_1 B_0(1+r) + \theta_2 S_0(1+\rho^*)) \to \min_{u,d,p} . \qquad (5)$$

In the problem (5) the resulting portfolio depends on u and d. In practice the problem could be solved numerically by using empirical distributions.

Further we assume that $R(y_1, y_2) = (y_1 - y_2)^2$. Then

$$R(f(S_1), \theta_1 B_0(1+r) + \theta_2 S_0(1+\rho^*)) =$$
$$= (f(S_1) - f(S_1^*) - \theta_2 S_0(\rho - \rho^*))^{\ 2} =$$

$$=\left[f(S_0(1+\rho) - f(S_0(1+\rho^*) - \frac{f(S_0(1+u)) - f(S_0(1+d))}{u-d}(\rho-\rho^*) \right]^2.$$

We will look for the random variable ρ^* in the form

$$\rho^* = \begin{cases} u, & if \quad \rho \ge c, \\ d, & if \quad \rho < c. \end{cases} \tag{6}$$

Thus the problem (5) in our case is to find such ρ^* of the type (6), which minimizes

$$E\left[f(S_0(1+\rho) - f(S_0(1+\rho^*) - \frac{f(S_0(1+u)) - f(S_0(1+d))}{u-d}(\rho-\rho^*) \right]^2. \tag{7}$$

3. **Numerical results**

Having the observations of the $(B_0^n, S_0^n, B_1^n S_1^n)$ in the discrete time moments $n = 1,2,...,N$ and assuming that $\rho_n = \frac{S_1^n - S_1^{n-1}}{S_1^{n-1}}$ are independent identically distributed random variables that have the same distribution as ρ we can minimize empirical estimator of (7) numerically.

The problem to be solved is

$$\frac{1}{N}\sum_{n=1}^{N}\left[f(S_0^n(1+\rho_n) - f(S_0^n(1+\rho_n^*) - \frac{f(S_0^n(1+u)) - f(S_0^n(1+d))}{u-d}(\rho_n - \rho_n^*) \right]^2 \to$$

$$\to \min_{u,d,c} \tag{8}$$

where

$$\rho_n^* = \begin{cases} u, & if \quad \rho_n \ge c, \\ d, & if \quad \rho_n < c. \end{cases}$$

The no- arbitrage condition gives us the constrains $u < 0$ and $d > 0$. Remind, that we assume $B_n^0 = B_n^1 = 1,\ n = 1,2,..,N$.

Here we assume that a derivative security is a 1- period European call option. Then

$$f(S_1) = \max(S_1 - K, 0),$$

where K-is the strike price of the call. We have performed the calculations for two choices of K. For the first one

$$K^1 = K_n = S_0^n(1+\bar{\rho}), \quad n = 1,2, \dots, N,$$

$$\bar{\rho} = \frac{1}{N}\sum_{n=1}^{N} \rho_n ,$$

for the second :

$$K^2 = \frac{1}{N}\sum_{n=1}^{N} S_0^n .$$

From (1)-(2) for the first choice of K the replicating portfolio approximation is given by the equalities

$$\theta_1^n = \frac{\max(S_0^n(1+d^*) - S_0^n(1+\overline{\rho}),0)(1+u^*) - \max(S_0^n(1+u^*) - S_0^n(1+\overline{\rho}),0)(1+d^*)}{u^* - d^*},$$

$$\theta_2^n = \frac{\max(u^* - \overline{\rho}),0) - \max(d^* - \overline{\rho}),0)}{u^* - d^*}.$$

Here u^* and d^* are the solutions of (8).We can note that θ_2^n does not depend on n.

For the second choice of K

$$\theta_1^n = \frac{\max(S_0^n(1+d^*) - K^2,0)(1+u^*) - \max(S_0^n(1+u^*) - K^2,0)(1+d^*)}{u^* - d^*},$$

$$\theta_2^n = \frac{\max(S_0^n(1+u^*) - K^2,0) - \max(S_0^n(1+d^*) - K^2,0)}{u^* - d^*}.$$

For the numerical illustration we consider the daily data of 3 Russian stocks SBER, GMKH, EERS since 01.04. 2003. We use N=50, and a derivative security is 1-month European call option.

In Table 1 we can see the parameters of the $(B, S^*)-$ market with the strike price K^1. In the sixth column Error is the square root of the minimum value of (8). This is the mean square error of the payoff function approximation. In Figures 1-3 the values of θ_1^n are shown

Table1

Stock	d^*	u^*	c^*	q_2	Error	Coefficient of variation	θ_2^n
SBER	-0,004843	0,104167	0,02457	0,044424	2.447484	0,478456	0.577214
GMKH	-0.032258	0.078838	0.071584	0.290361	0.484335	0.758302	0.390904
EERS	-0.027388	0.145867	0.042231	0.158087	0.003529	0.751422	0.555813

Figure 1

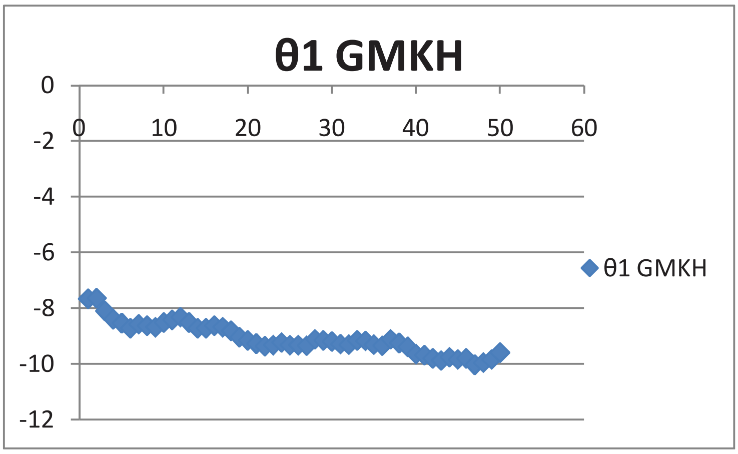


Figure 2

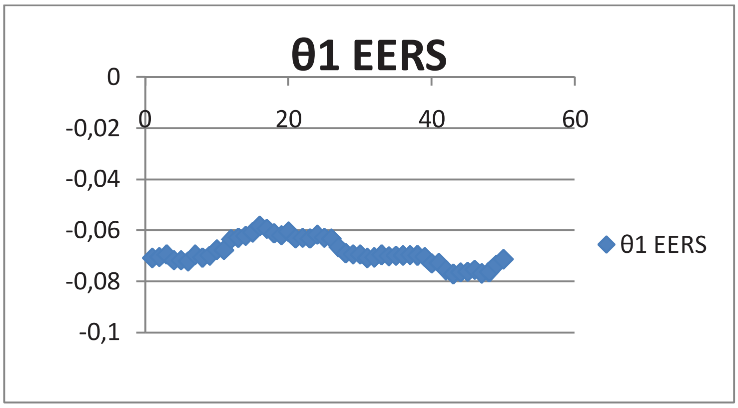
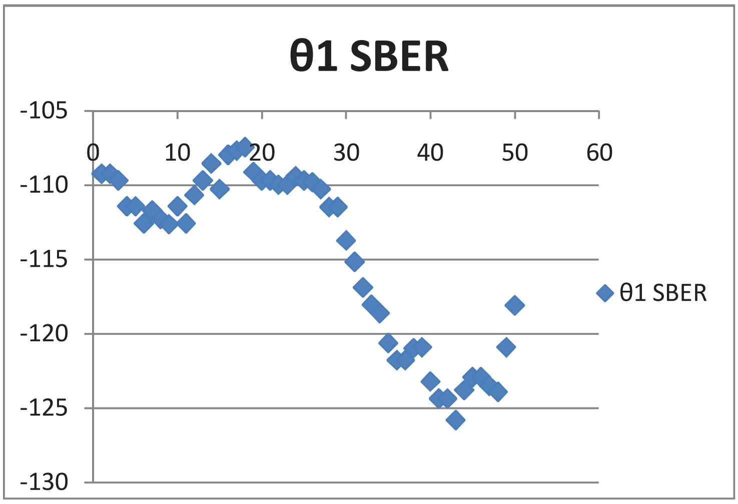


Figure 3



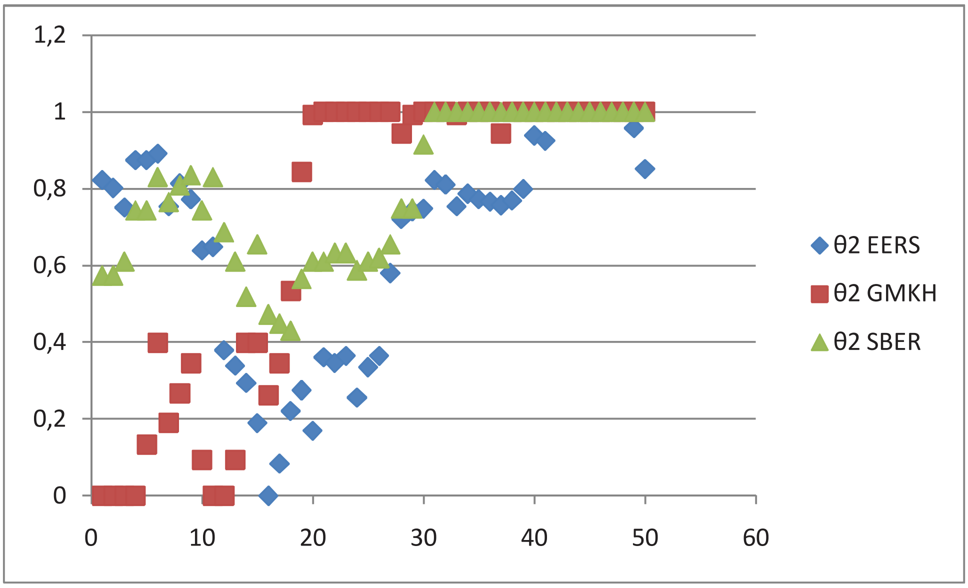
In Table2 the parameters of $(B,S^*)-$ market for $K=K^2$ are given

Table 2

Stock	d^*	u^*	c^*	q_2	Error	Coefficient of variation
SBER	-0,000927	0,120419	0,006835	0,007641	1.34778	0,147891
GMKH	-0,0082988	0,075051	0,071584	0,099565	0,226183	0,29262
EERS	-0.07635	0.17636	-0.0735	0.302113	0.003561	0.533157

In Figure 4 θ_2^n are shown for $K=K^2$

Figure 4



Figures 5-7 illustrate the dependence θ_1^n on *n*.

Figure 5

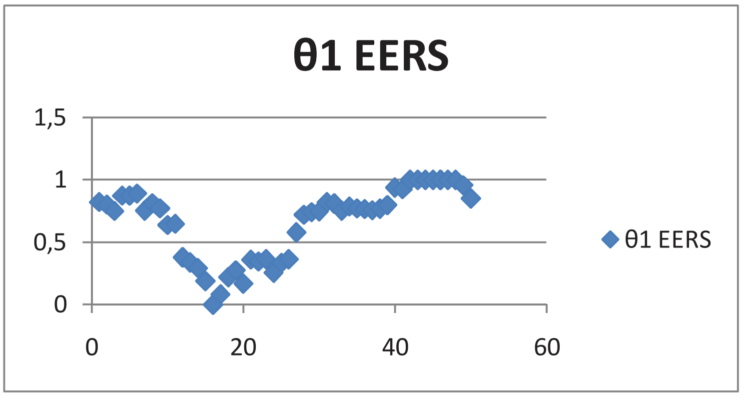


Figure 6

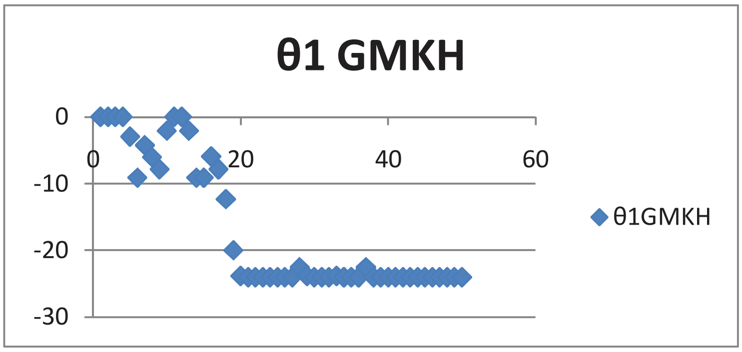
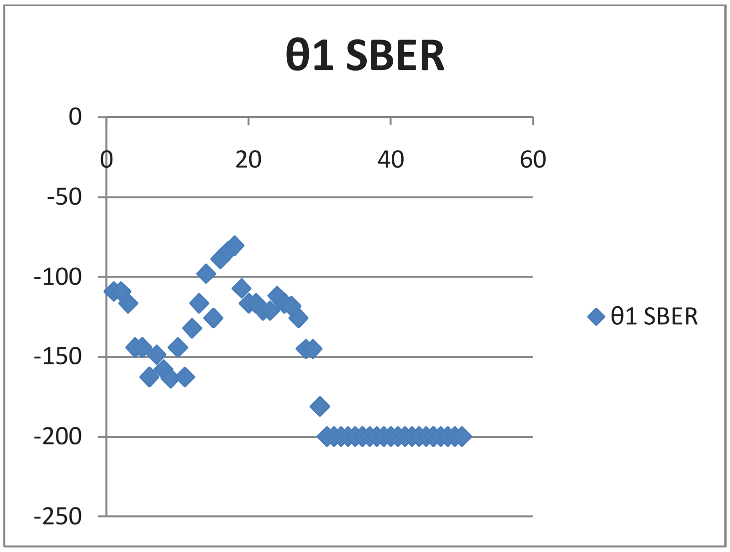


Figure 7



The values of the variation coefficients give a possibility to make outlines on the adequacy of the model under consideration. We see that the best result is for the SBER and GMKH stocks where $K = K^2$.

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