ASYMMETRIC AND COMPLEX RISK MEASURES

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- GIORGIO SZEGÖ: «Since its birth as an independent branch of social sciences, finance has witnessed three major revolutions:

  - mean-variance, 1952-56

  - continuous time models, 1969-73

  - risk measures, 1997-»

(Risk Measures for the 21st Century, John Wiley & Sons, 2004)

AGENDA

- Appropriate profitability characteristics
- Asymmetry in risk measurement for financial markets
- The risk measures application to optimal investment portfolio forming
- Asymmetric distortion risk measures
- Other asymmetric risk measures
- Complex quartile risk measures
- Complex index risk measures
- Conclusion

ASYMMETRY IN RISK MEASUREMENT FOR FINANCIAL MARKETS

- *Kahneman* and *Tversky's* prospect theory (*Econometrica*. 1979; 47: 263-291): there are a negligible number of people, wishing to participate in a lottery where equal profit and loss have the same probabilities.
- Failure stimulates activity of a company and can lead to some success in future. At the same time, success can weaken the successful companies.

ASYMMETRY IN RISK MEASUREMENT FOR FINANCIAL MARKETS

Applicability of the asymmetric risk measures was mentioned by

Kijima M., Ohnishi M. (*Annals of Operations Research*. 1993; 45: 147-163)

Gul F. (*Econometrica*. 1991; 59: 667-686)

Krzemienowski A. (*Ann Oper Res*. 2009; 165: 67–95)

APPROPRIATE PROFITABILITY CHARACTERISTICS

For an application of asymmetric risk measures such stock return rate should be used that have a set of values symmetric relative to zero and zero must correspond to a case of stock price stability (c_n is a stock price at the n-th day).

$$\chi_1(n) = \ln \frac{c_{n+1}}{c_n} \in (-\infty, \infty) \qquad \chi_2 = \frac{c_{n+1} - c_n}{c_{n+1} + c_n} \in [-1, 1)$$

THE RISK MEASURES APPLICATION TO THE OPTIMAL INVESTMENT PORTFOLIO FORMING

- Portfolio $U = (u_1, \ldots, u_n)$.
- u_i - a share of the means spent on the *i*-th kind of stocks, i.e. $u_i \geq 0$, $\sum_{i=1}^{n} u_i = 1$.
- T - time interval (long enough) for which stock price is observed.
- τ - time interval (short), following the interval T.

THE RISK MEASURES APPLICATION TO THE OPTIMAL INVESTMENT PORTFOLIO FORMING

Ψ_r – a family of risk measures depending on a vector parameter $r \in R$.

$\Psi^*_{r,T}(U)$ – the risk measure which is calculated for historical samples for portfolio's U profitability on the period T.

$\varphi_\tau(U)$ – some characteristic of the portfolio U effectiveness for an interval τ.

THE RISK MEASURES APPLICATION TO THE OPTIMAL INVESTMENT PORTFOLIO FORMING

To develop the most effective risk measures the following two – step optimization problem must be solved

$$U_T(r) = \arg\min_U \Psi^*_{r,T}(U)$$

$$r^0(T,\tau) = \arg\max_{r\in R} \varphi_\tau\left(U_T\left(r\right)\right)$$

THE RISK MEASURES APPLICATION TO THE OPTIMAL INVESTMENT PORTFOLIO FORMING

It is expedient to select the set

$$R^*(T,\tau)=\left\{r\in R:\varphi_\tau\left(U_T(r)\right)>\alpha r^0(T,\tau)\right\}$$

where the number α is close to 1 (for example 0,95; 0,99). If this technique gives close sets for the various markets then it is possible to recommend the corresponding risk measures to use for portfolio forming. If the risk measure $\Psi_{r,T}(U)$ has the minimum value, one can predict high enough income on the subsequent short time interval.

ASYMMETRIC DISTORTION RISK MEASURES

- Distortion risk measures introduced by Wang (*ASTIN Bulletin*. 1996; 26: 71-92) , are based on the Choquet's integral construction
- Asymmetric distortion risk measure (ADRM) were introduced by Sereda, Bronshtein, Rachev, Fabozzi, Sun, Stoyanov (*The Handbook of Portfolio Construction: Contemporary Applications of Markowitz Techniques, ch.25)* Springer-Verlag, 2010.

ASYMMETRIC DISTORTION RISK MEASURES

DEFINITION

Let $\bar{g}(t) = \left(g_1(t), g_2(t)\right)$ $g_i : [0,1] \rightarrow [0,1]$
be a pair of non-decreasing functions.

$$\Psi_{\bar{g}}(X) = \int_{-\infty}^{0} \left[1 - g_1\left(\bar{F}_X(t)\right)\right] dt - \int_{0}^{\infty} g_2\left(\bar{F}_X(t)\right) dt$$

where $\bar{F}_X(t) = 1 - F_X(t)$
is the additional distribution function of X=X(U) (may be both χ_1 and χ_2).
Standard distortion risk measures correspond to the case $g(t)$=$g_1(t)$=$g_2(t)$.

ASYMMETRIC DISTORTION RISK MEASURES

PROPERTIES

- ADRM doesn't depend on a X itself but only on its underlying distribution.
- $\Psi_{(t,t)}(X) = -\mathrm{E}[X]$.
- ADRM is positive for the guaranteed losses and negative for the guaranteed returns.
- ADRM isn't additive in the general case.

ASYMMETRIC DISTORTION RISK MEASURES

PROPERTIES

- Random variables are comonotonic if the increasing of one involves the increasing of the other variable. ADRM is additive if X, Y are comonotonic which always take on the values of the same signs.
- ADRM is monotonous: if almost surely $X \leq Y$, then $\Psi_{\bar{g}}(X) \geq \Psi_{\bar{g}}(Y)$.

ASYMMETRIC DISTORTION RISK MEASURES

PROPERTIES

- ADRM is positively homogeneous: if $\lambda \geq 0$ then $\Psi_{\bar{g}}(\lambda X) = \lambda \Psi_{\bar{g}}(X)$.
- The function $\Psi_{\bar{g}}(X + a)$ of a determined variable a is continuous and non-increasing.
- If $\bar{g}^*(t) = \left(1 - g_2(1-t), 1 - g_1(1-t)\right)$ then $\Psi_{\bar{g}^*}(-X) = -\Psi_{\bar{g}}(X)$.

ASYMMETRIC DISTORTION RISK MEASURES

SOME EMPIRICAL RESULTS

- The pairs of functions

$$\overline{g}_1(t) = (t^k, t^s),\ \overline{g}_2(t) = (1-(1-t)^k, 1-(1-t)^s)$$

  were examined
- 7 indexes (ASX200, XAX, BSE_SENTEX, BVSP, CAC40, CSI200, FTSE EUROTOP 100), T=364 days (one year), τ=2 days,
- enumeration of portfolios with step 0,17 (the total number equals 924), enumeration of pairs $(k,s)\in(0,3]\times(0,3]$ with step 0,2 (the total number is 225).

ASYMMETRIC DISTORTION RISK MEASURES

SOME EMPIRICAL RESULTS

- 51 pairs (k,s) for $\overline{g}_1$ and 45 pairs for $\overline{g}_2$ are close to optimal

- $k<s$ in 6 cases, $k=s$ in 2 cases, $k>s$ in 43 cases for $\overline{g}_1$
- $k<s$ in 25 cases, $k=s$ in 1 case, $k>s$ in 19 cases for $\overline{g}_2$

SOME OTHER ASYMMETRIC RISK MEASURES

PEDERSEN-SATCHELL'S ASYMMETRIC RISK MEASURE

Symmetrical variant was defined in "The Geneva Papers on Risk and Insurance Theory" 1998; 23: 89–117.

Asymmetrical version depends on six numerical and two functional parameters.

SOME OTHER ASYMMETRIC RISK MEASURES

ASYMMETRIC ORLICH TYPE RISK MEASURES

Orlich type risk measures were introduced

by J. Haezendonck, M.J. Goovaertc (*Insurance: Mathematics and Economics*. 1982; 1: 41-53) for positive risks,

by M.J.Goovaertc, R.Kaas, J.Dhaene, Q.Tang (*Insurance: Mathematics and Economics*. 2004; 34/3; 506-516.) the symmetrical extension.

SOME OTHER ASYMMETRIC RISK MEASURES

ASYMMETRIC ORLICH TYPE RISK MEASURES

Let function Φ: $(-\infty,\infty)\to(-\infty,\infty)$ (generalized normalized Jung function) have the properties:

- $\Phi(0)=0$, $\Phi(1)=1$, $\Phi(-1)=-1$,
- is increasing,
- is convex on a positive semiaxis and concave on a negative.

SOME OTHER ASYMMETRIC RISK MEASURES

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SOME OTHER ASYMMETRIC RISK MEASURES

ASYMMETRIC ORLICH TYPE RISK MEASURES

Risk measure is defined by the equation

$$E\left[\Phi\left(\frac{X}{\Psi(X)}\right)\right]=\alpha\in(0,1].$$

The measure depends on one numerical and one functional parameter.

Function Φ is odd in standard case.

SOME OTHER ASYMMETRIC RISK MEASURES

ASYMMETRIC SPECTRAL RISK MEASURES

- Spectral risk measures were defined by C. Acerbi (*Journal of Banking and Finance*, *2002*, **26**, 1505-1518).
- Asymmetric version.

$\varphi^{+}(u), \varphi^{-}(u)$ - non-increasing functions such as

$$\varphi^{+}(0) = \varphi^{-}(0) = 1; \varphi^{+}(1) = \varphi^{-}(1) = 0.$$

SOME OTHER ASYMMETRIC RISK MEASURES

ASYMMETRIC SPECTRAL RISK MEASURES

- $q_u(X) = F_X^{-1}(u)$ -u-quantile of distribution (in continuous case),
- $p^* = P[X < 0]$
- $\Psi(X) =$

$$= -A\left[\int_0^{p^*} \varphi^-(u) q_u(X) du + \int_{p^*}^{1} \varphi^+(u) q_u(X) du\right]$$

SOME OTHER ASYMMETRIC RISK MEASURES

RISK MEASURES BASED ON UTILITY FUNCTIONS

Generalized smooth utility functions $f(t)$ have the properties (S-shape function):

- $f(0)=0$,
- is increasing,
- is concave when $t \geq 0$ and convex when $t \leq 0$,
- is normalized: $f'(0)=1$.

SOME OTHER ASYMMETRIC RISK MEASURES

RISK MEASURES BASED ON UTILITY FUNCTIONS

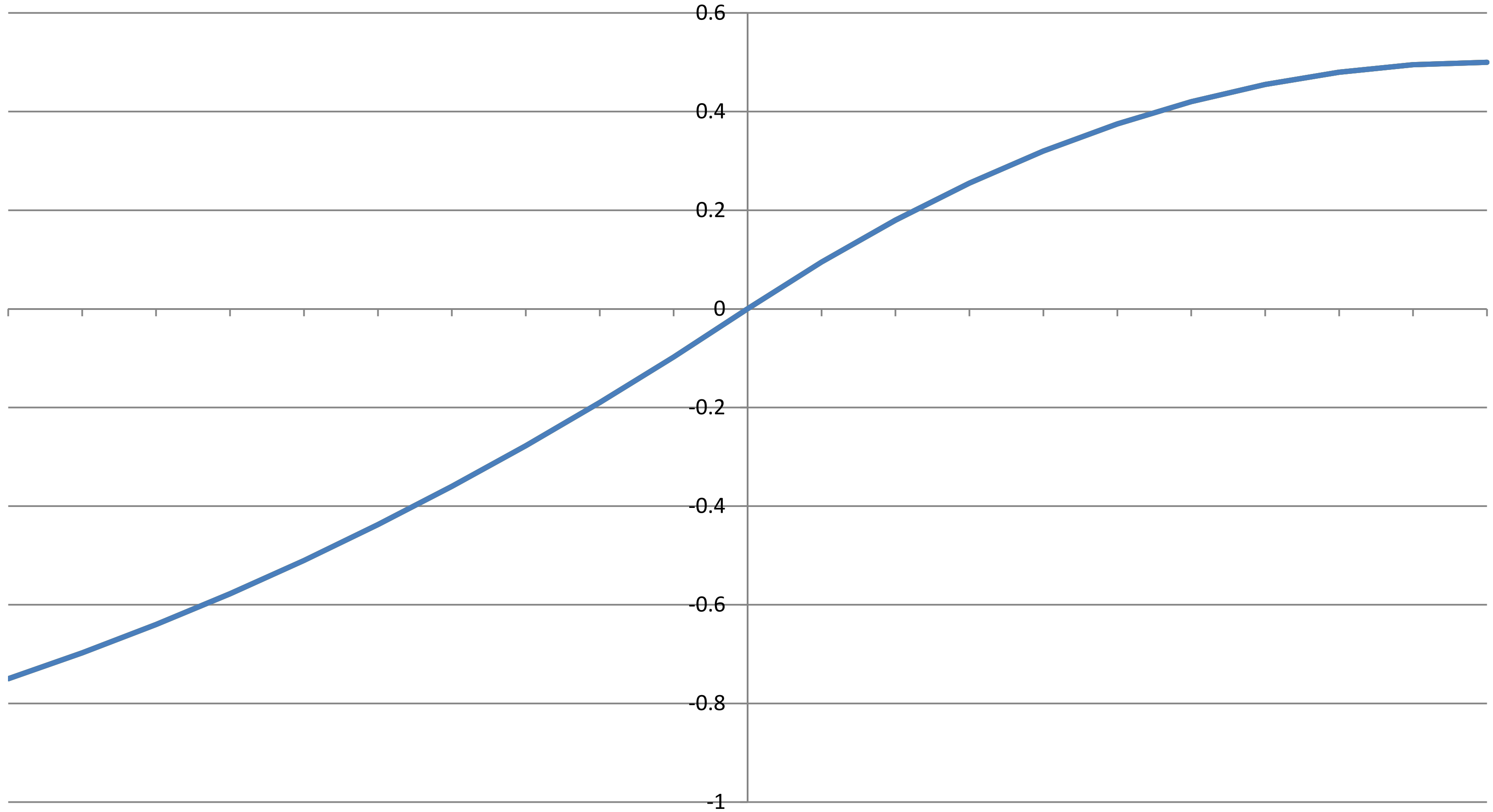
We defined quadratic, cubic and exponential functions.

For example

$$f(t) = \begin{cases} -a^{+}t^{2} + t & \text{if} \quad t \geq 0 \\ a^{-}t^{2} + t & \text{if} \quad t \leq 0 \end{cases}$$

$a^{+}, a^{-} \in (0, 1/(2 * \max\{t\})]$.

SOME OTHER ASYMMETRIC RISK MEASURES



SOME OTHER ASYMMETRIC RISK MEASURES

RISK MEASURES BASED ON UTILITY FUNCTIONS

$$\Psi(X) = -\int_{-p}^{p} f(t)\,dF_X(t)$$

$(-p,p)$ is the value range of profitability characteristic

COMPLEX QUANTILE RISK MEASURES

$\mathrm{VaR}_\alpha^-[X]$ is distribution of X quantile.

α is the probability of the event "X is smaller than $\mathrm{VaR}_\alpha^-[X]$".

α usually takes values 0,1; 0,05; 0,01

$\mathrm{CVaR}_\alpha^-[X]$ is a conditional mean of X if X is lower than or equal to $\mathrm{VaR}_\alpha^-[X]$.

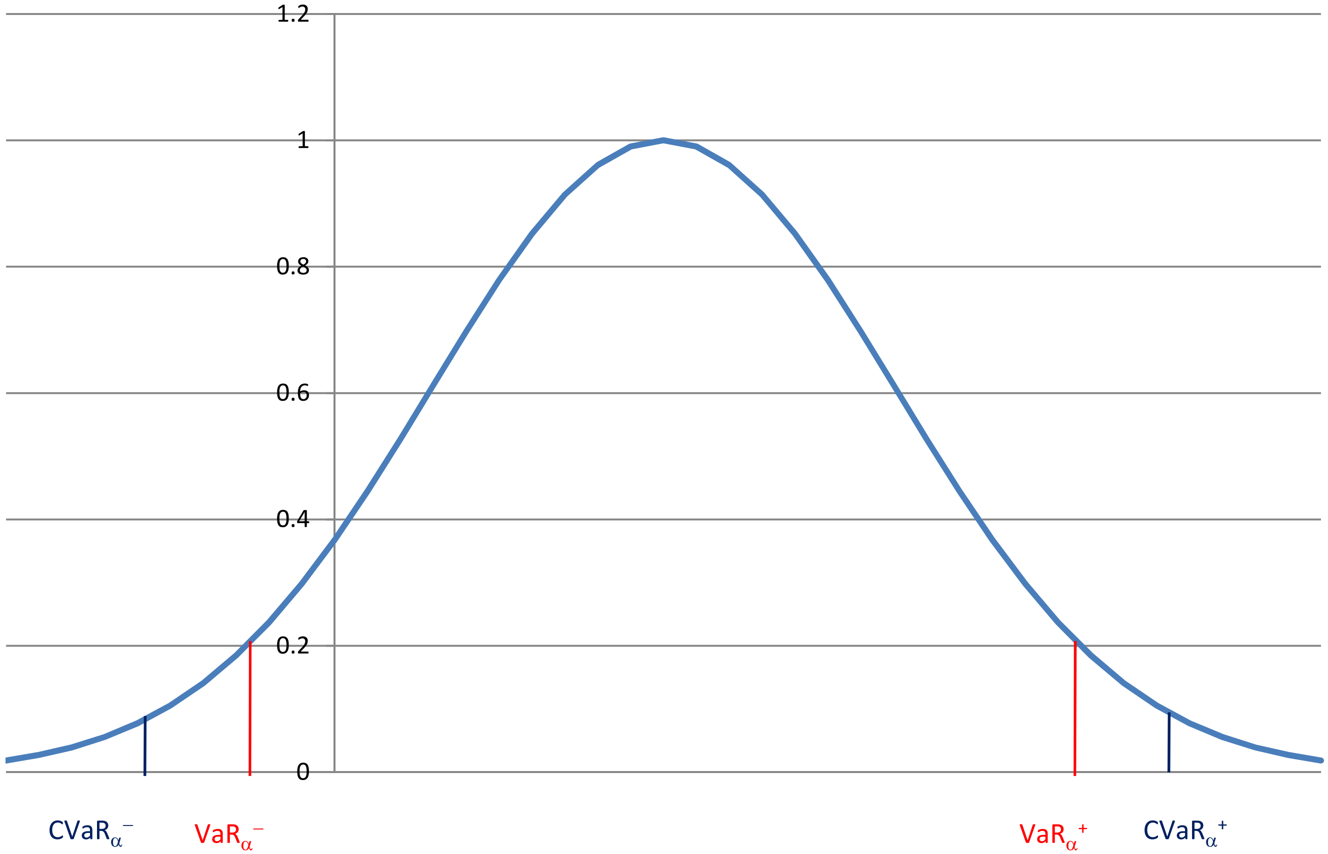
Both measures take into consideration only the weight of the left tail of the distribution.

COMPLEX QUANTILE RISK MEASURES

- It is reasonable to consider the weight of the right tail
- $\mathrm{VaR}_\alpha^+[X]$ – $(1-\alpha)$-quantile of the distribution X
- $\mathrm{CVaR}_\alpha^+[X]$ – conditional mean of X if X is more than or equal to $\mathrm{VaR}_\alpha^+[X]$.

  Such approaches were used by J.Dhaene et al (Stochastic Models. 2006. 22. 573–606), S.Rachev et al (International Journal of Theoretical and Applied Finance. 2008. v.11. No.1. 19-54

COMPLEX QUANTILE RISK MEASURES



COMPLEX QUANTILE RISK MEASURES

$$VaR_\alpha^-(X) = x_{[\alpha n]},\ CVaR_\alpha^-(X) = \frac{\sum_{i=1}^{[\alpha n]} x_i}{[\alpha n]},$$

$$VaR_\alpha^+(X) = x_{n-[\alpha n]},\ CVaR_\alpha^+(X) = \frac{\sum_{i=n-[\alpha n]+1}^{n} x_i}{[\alpha n]}.$$

COMPLEX QUANTILE RISK MEASURES

$$M_1(X : \alpha, \beta, k) = kVaR_{\alpha}^{-}(X) + (1-k)CVaR_{\beta}^{-}(X),$$

$$M_2(X : \alpha, \beta, k) = kVaR_{\alpha}^{+}(X) + (1-k)CVaR_{\beta}^{+}(X),$$

$$M_3(X : \alpha, \beta, k) = k \cdot 10^5 VaR_{\alpha}^{-}(X) + (1-k)VaR_{\beta}^{+}(X),$$

$$M_4(X : \alpha, \beta, k) = k \cdot 10^5 CVaR_{\alpha}^{-}(X) + (1-k)CVaR_{\beta}^{+}(X).$$

COMPLEX QUANTILE RISK MEASURES

SOME EMPIRICAL RESULTS

- Were analyzed three sets of Russian stocks for T =2 years and τ =a week, a month, half a year, a year with different lags, $k \in [0, 1]$ with the step 0,1; $\alpha, \beta \in [0,02; 0,1]$ with the step 0,01 (891 variants).
- It was established that
  - the results are better for risk measure M_2 if τ is a month, $k \approx 0,7$, $\alpha \approx \beta \approx 0,05$,
  - the results are better for risk measure M_3 if τ is a half year, $k \approx 0,5$, $\alpha \approx \beta \approx 0,05$,
  - if τ is a week and a year none of these measures is a winner.

COMPLEX INDEX RISK MEASURES

$$M_1(\alpha, \beta) = \frac{VaR_\alpha^-(X)}{VaR_\alpha^+(X)} - \beta\gamma_1(X)$$

$$M_2(\alpha, \beta) = \frac{CVaR_\alpha^-(X)}{CVaR_\alpha^+(X)} - \beta\gamma_1(X)$$

COMPLEX INDEX RISK MEASURES

$$\gamma_1(X) = \frac{E\left[\left(X - E[X]\right)^3\right]}{\sigma^3}$$

- skewness coefficient

the approach is based on some ideas of S. Rachev, C. Menn, F. Fabozzi (*Fat-Tailed and Skewed Asset Return Distribution*. Wiley Finance, 2005).

COMPLEX INDEX RISK MEASURES

SOME EMPIRICAL RESULTS

- For comparison Sharpe index

$$M_3 = -\frac{E[X]}{\sigma[X]}$$ was used.

COMPLEX INDEX RISK MEASURES

SOME EMPIRICAL RESULTS

1. For Russian and World markets the results are very close (optimum parameters are α: [0,18; 0,22],

   β: [1; 2] for M_1; α: [0,15; 0,35], β: [2; 4] for M_2.

2. The usage of index measures based on the measure *CVaR* and skewness, generate more effective portfolio than for Sharpe index and for the measure based on *VaR*.

CONCLUSION

A few new risk measures are introduced.
Some of them take into account asymmetry of reaction of profits and losses, other are complex.
Numerical experiments on real samples show the efficiency of some of them for portfolio forming.

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