Teaching life insurance mathematics: from the l_x's to the Risk Management approach

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Abstract

We focus on a modern approach to life insurance mathematics, properly allowing for various risk sources. The prevailing deterministic principles of traditional actuarial calculations affected, for a long time, many life insurance courses. Conversely, the presence of risks arising from mortality, financial issues and policyholders' behaviour should be properly accounted for, and suggest appropriate contents of life insurance courses. As regards mortality, an appropriate setting should focus on the insureds' random lifetimes and the related survival function. Thus, the expression of the random present value of insured benefits as a function of the residual lifetime constitutes the starting point of a sound stochastic approach. Modern actuarial textbooks adopt this approach, thus allowing for the "process risk", namely the risk arising from fluctuations of individual lifetimes around the relevant expectations, and hence for the volatility of insurer's payouts. It is usually assumed that a given survival function provides an appropriate description of random lifetimes, and hence of portfolio results. However, some degree of uncertainty may affect the choice of the survival function. This risk is usually referred to as the "uncertainty risk", namely uncertainty in the level of mortality and/or its future trend. An example is given by the longevity risk, arising from uncertainty in future mortality trends, which has a dramatic importance in large portfolios (or pension funds) providing lifelong benefits. Actually this risk cannot be lowered by increasing the portfolio size, since it concerns the portfolio as an aggregate. In this paper, we stress the importance of stochastic actuarial modelling. Further, advantages provided by a modern risk-management approach, ranging from risk identification to the choice of appropriate actions, are addressed. Focussing on life annuities, we stress several points in favour of emphasis on risk when dealing with life insurance technical problems and teaching life insurance mathematics.

Keywords

Actuarial education, Actuarial modelling, Stochastic mortality, Process risk, Uncertainty risk, Longevity risk, Life annuities, Living benefits

1 Introduction

The awareness of advantages provided by "large" portfolio sizes can be traced back to the end of the 18th century. In 1786 Johannes Tetens addressed the analysis of mortality risk inherent in insurance portfolios. In his analysis, the evidence of the role of $\sqrt{n}$ in determining the riskiness of a portfolio, where n denotes the number of policies in the portfolio itself, emerges. In particular, as pointed out by Haberman (1996), Tetens showed that the risk in absolute terms increases as the portfolio size n increases, whereas the risk in respect of each insured decreases in proportion to $\sqrt{n}$. In a modern perspective, Tetens' ideas constitute a pioneering contribution to individual risk theory and insurance risk management.

Teten's findings about the "small" variability in the insurer's payout when the portfolio size is reasonably "large" justify (at least to some extent) the deterministic approach to life insurance calculations. It is worth noting that, even if the so called biometric functions, e.g. q_x, l_x, and so on, involved in actuarial models, are correctly meant as probabilities, expected values, etc., when these quantities are only used to calculate expected values of benefits and premiums we actually should refer to a deterministic approach, as no measure of riskiness is allowed for.

Obviously, approaches adopted and topics dealt with in many life insurance mathematics courses have been, for a long time, heavily affected by the prevailing "deterministic principles" of actuarial calculations.

Conversely, a stochastic approach to actuarial calculations requires focussing on random variables and related probability distributions. In particular, an appropriate tool is provided by the random residual lifetime of an individual aged x, T_x, and the relevant probability distribution, often assigned in terms of the survival function (referred to the random total lifetime T_0), $S(t) = \mathbb{P}[T_0 > t]$. The expression of the random present value (e.g. at policy issue) of insured benefits as a function of the residual lifetime T_x (x being the age at policy issue) is due to B. de Finetti (see de Finetti (1950) and de Finetti (1950)) and Sverdrup (1952), and constitutes the starting point of a sound stochastic approach based on individual lifetimes (see also Pitacco (2004b)).

Modern actuarial textbooks adopt this approach, and hence life insurance products are presented in a rigorous framework, allowing, at least to some extent, for riskiness inherent in cash flows of benefits and premiums. See, for example, Gerber (1995), Bowers et al. (1997), Gupta and Varga (2002), and Promislow (2006).

A stochastic approach as described above allows for the so called "process risk" only, namely the risk arising from randomness (or "volatility") of the individual lifetimes around the relevant expectations, and hence for random fluctuations of the insurer's payout. Actually, the probabilistic structure (e.g. the survival function $S(t)$) is assumed to provide an appropriate description of randomness in policyholders' lifetimes and hence in portfolio results. However, some degree of uncertainty may affect the choice of the survival function. If a law (e.g. Gompertz, Makeham, Thiele, etc.) has been chosen to describe the age pattern of mortality, uncertainty may concern the type of the function or its parameter values. In the former case, the "model risk" arises, while in the latter the "parameter risk" is involved. In both cases, the risk is usually referred to as the "uncertainty risk", the term "uncertainty" referring to the representation of the level of a phenomenon (say, the mortality) and/or its future trend.

While the process risk should be carefully accounted for in particular when small insurance portfolios (or pension funds) are addressed, the uncertainty risk has a dramatic importance especially in relation to large portfolios of lifelong benefits (or pension funds). Actually this risk cannot be lowered by increasing the portfolio size, since it concerns the portfolio as an aggregate risk. An important example of uncertainty risk is given by the "longevity risk" arising from uncertainty in future mortality trends, and possibly leading to underestimation of insurer's liabilities when benefits consist in life annuities.

Let us focus on some issues emerging from the introductory ideas presented above. Mortality constitutes a "cause" of risk. Its "components" are the process risk and the uncertainty risk (as well as the catastrophe risk). Mortality risk impacts on several portfolio "results", for example: cash flows, profits, shareholders'capital, etc. The severity of the impact depends on various risk "factors". Example of risk factors are: the portfolio size, the statistical distribution of sums insured (the higher the dispersion, the heavier the impact of mortality random fluctuations), and so on.

Of course, other causes of risk also affect the results of a life insurance portfolio, for example: the market (or investment) risk, the expense risk, the credit risk, the lapse risk, etc. For each cause, risk components and factors can be singled out.

While thinking in the above terms, we are moving towards a risk-management oriented approach to life insurance technical problems. Actually, the "identification" of risks constitutes the first step of the Risk Management (RM) process.

Actuarial mathematics constitutes an early example of rigorous approach to risk assessment, and can be considered one of the first application of RM (see Tapiero (2004)) as it also aims at properly covering risks inherent in insurance portfolios, e.g. through an appropriate capital allocation. Nevertheless, a "formal" approach to life insurance technical problems inspired to RM principles, ranging from risk identification to product design and hedging techniques, is a very recent issue, not yet widely implemented (for example, see Panjer (2006)).

It is worth stressing, however, that the new framework for solvency assessment designed by Solvency 2 rules, basically relies on a RM approach to insurance business, in both the life and non-life area. The teaching of life insurance mathematics should then allow for this new trend. Clearly, the range of topics addressed and the depth of the analysis strictly depend on time allocated in teaching syllabuses to life insurance technical issues. Anyhow, a RM oriented approach should concern university courses devoted to insurance techniques in both actuarial and economic curricula.

This paper is organized as follows. In Sect. 2 the basic structure of the RM process is briefly described, and its applications to the life insurance business are sketched. Section 3 addresses life annuities and constitutes the core of the paper, as it presents a comprehensive example of a RM perspective for dealing with life insurance problems. Section 4 provides a summary of various approaches to risk assessment. Mortality risks also affect insurance products in the area of health insurance; some related aspects are addressed in Sect. 5. Final remarks in Sect. 6 conclude the paper.

The paper is mostly based on research work recently performed by the author (Department of Applied Mathematics at the University of Trieste) jointly with Annamaria Olivieri (Department of Economics at the University of Parma), and teaching experience in both undergraduate and graduate courses. Various topics concerning stochastic mortality in life annuities have been dealt with by Annamaria Olivieri and the author in several CPD initiatives, and in particular (together with Michel Denuit and Steven Haberman) in the Groupe Consultatif Summer School on "Modelling mortality dynamics for pensions and annuity business", held in Trieste, 2005, and Parma, 2006.

The paper is also based on the book by Pitacco et al. (2009), which the interested reader can refer to for a detailed presentation of various issues concerning in particular the longevity risk. Further, the book by Olivieri and Pitacco (2010) will provide a RM-based introduction to technical and financial aspects of the insurance business.

2 The Risk Management process

Figure 2.1 illustrates the basic steps in the RM process (left-hand side of the figure), and the corresponding life insurance implementation (right-hand side).

As regards the risk identification phase, guidelines can be drawn from several risk "classifications" proposed in the technical literature. A very interesting example is provided by IAA (2004).

As far as risk assessment is concerned, deterministic tools like the sensitivity analysis should first be used in teaching activities, in order to improve students' awareness in respect of the role of various variables defining the scenario of the insurance business. For example: investment yield, lapse rates, age-pattern of mortality, expenses, etc.

Stochastic analysis should then be carried out, by adopting both analytical tools (when practicable) and Montecarlo simulation procedures. The latter can be used, for example, in order to estimate the probability of default, and constitute a very powerful tool also having the appeal of a "constructive" approach to risk and uncertainty assessment. See also Sect. 4.

RM techniques correspond to portfolio strategies in RM application to the insurance business. In particular, when dealing with product pricing, we go back to the classical field of actuarial mathematics. Notwithstanding, a RM perspective suggests that, in a number of cases, the traditional equivalence principle, together with the implicit safety loading practice, should be replaced by more appropriate pricing principles explicitly accounting for riskiness. Although principles other than the equivalence principle already constitute a topic in most actuarial courses, the framework provided by RM first suggests to carefully look at all the components of an insurance product, placing a special emphasis on guarantees and options implying a high degree of risk.

Moreover, the RM approach to portfolio strategies suggests a comprehensive vision including various actions, e.g. pricing, capital allocation, risk transfers, etc. As regards in particular risk transfers, "alternative" risk transfers (ART) deserve particular attention in life insurance management, because of the increasing importance of securitization via mortality bonds and longevity bonds. From a teaching point of view, securitization issues offer an interesting opportunity for establishing a non-traditional link between insurance and capital markets.

3 A RM approach to life annuities

Great attention is currently devoted to the management of life annuity portfolios, both from a theoretical and a practical point of view, because of the growing importance of annuity benefits paid by private schemes. In particular, the progressive shift from defined benefit to defined contribution pension schemes increases the interest in life annuities with a guaranteed annual amount (see Pitacco et al. (2009)).

Life annuities provide an outstanding example of need for stochastic valuations and appropriate risk assessment. In this sense, properly dealing with life annuities constitutes an invaluable opportunity for a risk-management oriented teaching.

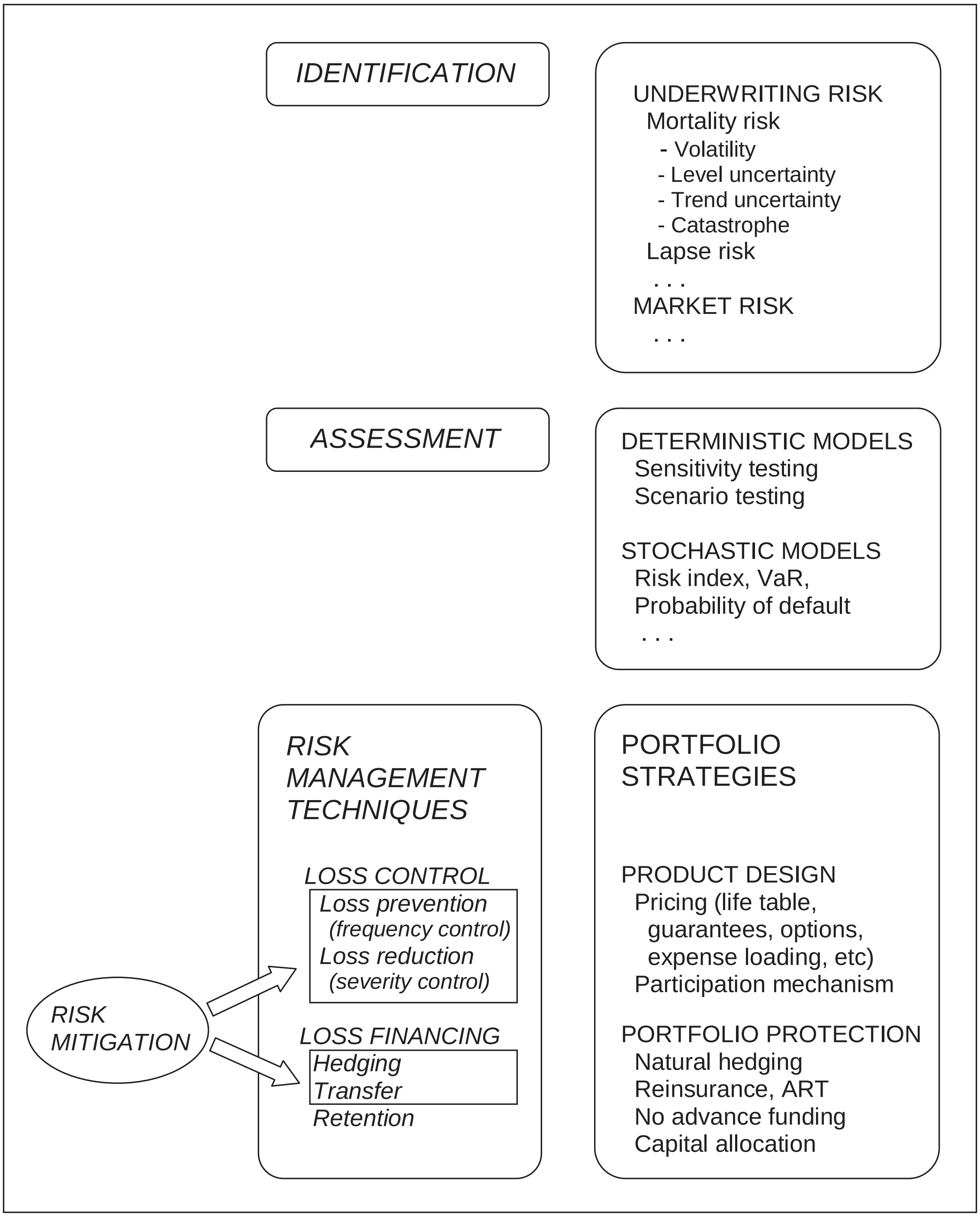

Figure 2.1: The Risk Management process

In what follows we illustrate the stochastic nature of life annuities through a sequence of steps, starting from the annuity-certain and progressively extending the domain of uncertainty. Advantages (and disadvantages) of life annuities compared to withdrawal processes, from the point of view of a retiree, can be clearly singled out.

3.1 Annuity certain versus life annuity

Assume that an amount S is available at a certain time, say at retirement. Denote this time with $t = 0$. In order to get her/his post-retirement income, the retiree withdraws from the fund at time t an amount b_t ($t = 1, 2, \dots$). Assume that the fund is managed by a financial institution which guarantees a constant annual interest rate i.

Denote with F_t the fund at time t, immediately after the payment of the annual amount b_t. Clearly:

$$F_0 = S \tag{3.1}$$

and

$$F_t = F_{t-1}(1 + i) - b_t \quad \text{for } t = 1, 2, \dots \tag{3.2}$$

The behavior of the fund throughout time obviously depends on the sequence of withdrawals $b_1, b_2, \dots$. In particular, assume that for all times t the annual withdrawal is equal to the annual interest credited by the fund manager, that is

$$b_t = F_t\, i \tag{3.3}$$

then, from Eq. (3.2) we immediately find

$$F_t = S \tag{3.4}$$

for all t, whence a constant withdrawal

$$b = S\, i \tag{3.5}$$

follows.

Conversely, if the constant withdrawal is greater than the interest, that is

$$b > S\, i \tag{3.6}$$

(as likely needed in order to obtain a reasonable post-retirement income) the withdrawal process can exhaust, sooner or later, the fund. The possible exhaustion depends, of course, on the retiree's lifetime.

From recursion (3.2) we have

$$F_0 > F_1 > \dots > F_t > \dots \tag{3.7}$$

and hence we can find a time m such that

$$F_m \geq 0 \text{ and } F_{m+1} < 0 \tag{3.8}$$

Clearly, the exhaustion time m depends on the annual amount b (and the interest rate i as well), as it can be easily seen from Eq. (3.2).

The sequence of m constant annual withdrawals b (with m defined by conditions (3.8), and possibly completed by the exhausting withdrawal at time $m + 1$) constitutes an annuity-certain.

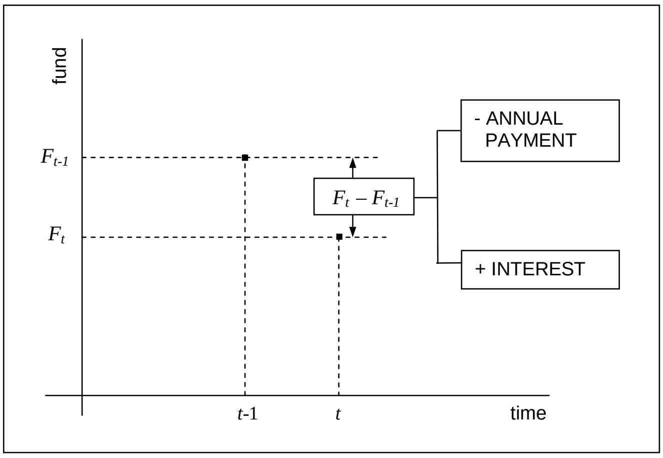

Figure 3.1: Annual variation in a fund providing an annuity-certain

Figure 3.1 summarizes the causes explaining the behavior of the fund throughout time. It is worth noting that, looking at the two causes of change in the fund level, we can single out two corresponding causes of risk borne by the financial institution providing the annuity:

– market risk, more precisely interest rate risk, as we have assumed that i is the guaranteed interest rate;

– liquidity risk, as the annual payment obviously requires cash availability.

Conversely, the risk related to the lifetime is obviously borne by the retiree, as the annuity payments cease at fund exhaustion.

Risks related to random lifetimes can be transferred from the annuitants to the annuity provider thanks to a different contractual structure, i.e. the life annuity. Technical features of life annuities can be introduced via a (very) traditional model, looking at mortality according to a deterministic approach.

3.2 Allowing for mortality: the deterministic approach

Consider the following transaction: an individual age x pays to the annuity provider (for example an insurer) an amount S to receive a (life) annuity consisting in a sequence of annual benefits b, paid at the end of every year while she/he is alive. Assume that the same type of annuity is purchased at time $t = 0$ by a given number, say l_x, of individuals all age x.

Let l_{x+t} denote an estimate (at time 0) of the number of individuals (annuitants) alive at age $x+t$ ($t = 1, 2, \dots, \omega - x$), out of the initial "cohort" of l_x individuals; ω denotes the age such that $l_\omega > 0$ and $l_{\omega+1} = 0$. The following (estimated) cash flows of the annuity provider are then defined:

(a) an income $l_x\, S$ at time 0;

(b) a sequence of outgoes $l_{x+t}\, b$ at time t, $t = 1, 2, \dots, \omega - x$.

Let V_t denote the fund pertaining to a generic annuitant at time t. In terms of V_t the fund of the annuity provider, given by $l_{x+t}\, V_t$, is defined for $t = 0$ as

$$l_x\, V_0 = l_x\, S \tag{3.9}$$

and, for $t = 1, 2, \dots \omega - x$, as follows:

$$l_{x+t}\, V_t = l_{x+t-1}\, V_{t-1}\,(1+i) - l_{x+t}\, b \tag{3.10}$$

From Eq. (3.10), we find the following recursion describing the evolution of the individual fund:

$$V_t = \frac{l_{x+t-1}}{l_{x+t}}\, V_{t-1}\,(1+i) - b \tag{3.11}$$

which can also be written as follows

$$V_t = V_{t-1}\,(1+i) + \frac{l_{x+t-1} - l_{x+t}}{l_{x+t}}\, V_{t-1}\,(1+i) - b \tag{3.12}$$

It is worth noting from (3.12) that the annual decrement of the individual fund can be split into three contributions (see Fig. 3.2) :

(a) a positive contribution provided by the interest $V_{t-1}\, i$;

(b) a positive contribution provided by the share of the reserves released because of the death of $l_{x+t-1} - l_{x+t}$ annuitants in the t-th year, the share being credited to the l_{x+t} annuitants alive at time t;

(c) a negative contribution equal to the annual amount b.

Contribution (b), which does not appear in the model describing the annuity-certain (see Fig. 3.1), is maintained thanks to a cross-subsidy among annuitants, i.e. the so-called "mutuality effect".

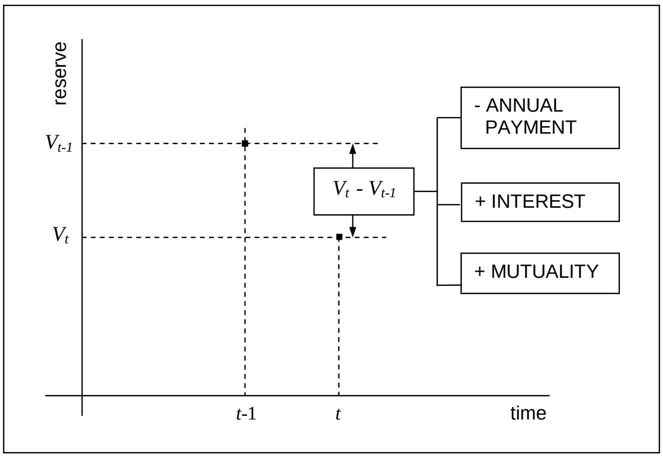

Figure 3.2: Annual variation in the mathematical reserve of a life annuity

To complete the (deterministic) life annuity model, the relation between the sum S and the annual payment b must be explicitly stated. Clearly the relation depends on the interest rate i and the estimated number of survivors l_{x+t}. For a given i and a given sequence $\{l_{x+t}\}$, we can solve Eq. (3.11) with respect to b (or S) when S (or b) has been assigned. We find

$$l_x\, S = b \sum_{t=1}^{\omega-x} l_{x+t}\, (1+i)^{-t} \tag{3.13}$$

and, referring to a single annuitant,

$$S = b \sum_{t=1}^{\omega-x} \frac{l_{x+t}}{l_x}\, (1+i)^{-t} \tag{3.14}$$

Some comments can help in understanding the features of the deterministic model. First, a point in favour of the model is that, in spite of its deterministic nature, the risk arising from random lifetimes can be clearly perceived. Indeed, from Eq. (3.12) we see that, if the actual lifetimes of the annuitants lead, in year t, to a number of deaths smaller than the estimated one (that is, $l_{x+t-1} - l_{x+t}$), the cross-subsidy mechanism cannot finance the annuity payments to the annuitants still alive at time t. In other words, contribution (b), which is required to maintain the individual fund V_t, should be partially funded, in this case, by the annuity provider. Conversely, a number of deaths greater than the estimated one leads to a provider's profit.

Secondly, Eq. (3.14) can be rewritten in "probabilistic" terms, since $\frac{l_{x+t}}{l_x}$ can be interpreted as the estimate of

$$\mathbb{P}[T_x > t] = {}_t p_x \tag{3.15}$$

so that we have

$$S = b \sum_{t=1}^{\omega-x} {}_t p_x\, (1+i)^{-t} \tag{3.16}$$

that is, the well known formula of Edmund Halley (for example, see Pitacco (2004d)). An alternative expression is provided by the formula of Jan de Witt

$$S = b \sum_{h=1}^{\omega-x} a_{\overline{h|}}\; {}_{h|1}q_x \tag{3.17}$$

(according to the usual notation). Clearly, the right-hand side of Eq. (3.17) represents the expected present value, or "actuarial value", of the life annuity:

$$S = b\; \mathbb{E}[a_{\overline{K_x|}}] \tag{3.18}$$

where K_x denotes the curtate random remaining lifetime of an individual aged x; with the usual symbol:

$$S = b\, a_x \tag{3.19}$$

Finally, the quantity V_t can be interpreted as the mathematical reserve of the life annuity, whose evolution throughout time is described by recursion (3.11).

It should be noted that, although Eqs. (3.16) and (3.17) involve probabilities, the model built up sofar is a deterministic model, as probabilities are only used to determine expected values.

Before shifting to a stochastic framework, it is worth noting that comparing the features of an annuity-certain to those of a life annuity provides some hints of interest under a teaching perspective. Indeed, the choice between an annuity-certain and a life annuity constitutes a problem of "personal risk management". An annual withdrawal just equal to the annual interest clearly does not consume the fund and hence complies with a strong bequest motivation. Annual withdrawals greater than the interest tend to exhaust the fund and, at the same time, charge the retiree with the risk arising from her/his random lifetime. Finally, the life annuity mechanism transfers this risk to the annuity provider, whilst no bequest possibility is left to the annuitant as the individual fund (reserve) available at the time of death is assigned to annuities still in force.

3.3 From deterministic to stochastic: the random present value of a life annuity

De Witt's Eq. (3.17) involves the random present value Y,

$$Y = a_{\overline{K_x}|} \tag{3.20}$$

of a life annuity, as shown by (3.18). The notation used in (3.20) was proposed by de Finetti (1950) and de Finetti (1957), in order to denote random present values of insured benefits. Other examples are as follows:

- for the whole life assurance: $Y = v^{T_x}$, and hence $\bar{A}_x = \mathbb{E}[v^{T_x}]$;
- for the endowment insurance: $Y = v^{\min(T_x,n)}$, and hence $\bar{A}_{x,\overline{n}|} = \mathbb{E}[v^{\min(T_x,n)}]$.

De Witt's formula uses the ${}_{h|1}q_x$'s which constitute the probability distribution of the curtate remaining lifetime. Thus, moments other than the expected value can be calculated, for example the variance of $a_{\overline{K_x}|}$.

When an annuity portfolio is addressed, the expected value and the variance of the total random present value of benefits can be calculated (provided that simplifying hypotheses for the calculation of the variance have been assumed, e.g. the stochastic independence of the annuitants' lifetimes). Assume that the portfolio is a cohort initially (at time 0) consisting of l_x annuitants. The annual payment is b for all annuitants. The rate of interest is assumed to be deterministic in any case.

The random present value at time 0 of benefits payable to annuitant j is given by

$$Y^{(j)} = b\, a_{\overline{K_x^{(j)}}|} \tag{3.21}$$

where $K_x^{(j)}$ is her/his curtate residual lifetime, and $a_{\overline{K_{x_0}^{(j)}}|}$ relies on a given interest rate i.

In particular, expected value and variance of the individual random present value of benefits, $Y^{(j)}$, can be calculated:

$$\mathbb{E}[Y^{(j)}] = b\; \mathbb{E}[a_{\overline{K_x^{(j)}}|}] \tag{3.22}$$

$$\mathbb{V}\mathrm{ar}[Y^{(j)}] = b^2\, \mathbb{V}\mathrm{ar}[a_{\overline{K_x^{(j)}}|}] \tag{3.23}$$

Let us denote by $Y^{(\Pi)}$ the random present value of future benefits for the portfolio. Clearly, it can be expressed as follows

$$Y^{(\Pi)} = \sum_{j=1}^{l_x} Y^{(j)} \tag{3.24}$$

Assume that the random variables $K_x^{(j)}$ are independent and identically distributed. The same assumptions then concern the random variables $Y^{(j)}$. Hence, the expected value and variance of $Y^{(\Pi)}$ are

$$\mathbb{E}[Y^{(\Pi)}] = l_x \, \mathbb{E}[Y^{(j)}] = l_x \, b \, a_x \tag{3.25}$$

$$\mathbb{V}\mathrm{ar}[Y^{(\Pi)}] = l_x \, \mathbb{V}\mathrm{ar}[Y^{(j)}] \tag{3.26}$$

Interesting insights into the nature and importance of the mortality risk can be obtained from the coefficient of variation of $Y^{(\Pi)}$, $r(l_x)$, defined as follows:

$$r(l_x) = \frac{\sqrt{\mathbb{V}\mathrm{ar}[Y^{(\Pi)}]}}{\mathbb{E}[Y^{(\Pi)}]} = \sqrt{\frac{1}{l_x} \frac{\mathbb{V}\mathrm{ar}[Y^{(j)}]}{\mathbb{E}^2[Y^{(j)}]}} \tag{3.27}$$

The index $r(l_x)$ allows us to investigate the effect of the (initial) portfolio size l_x on the overall riskiness. Equation (3.27) shows that, in relative terms, the riskiness of the portfolio decreases as l_x increases (according to Tetens' findings). In particular, we have

$$\lim_{l_x \to \infty} r(l_x) = 0 \tag{3.28}$$

This gives account of the common opinion that the larger is the portfolio, the less risky it is, since with higher probability observed values will be close to expected ones. We briefly call $r(l_x)$ the risk index.

Moving from the deterministic model presented in Sect. 3.2 to the stochastic model we have just described allows us to assess the risk of random fluctuations around expectations, namely the volatility. Further results can be obtained in this context; for example, the probability distribution of $Y^{(\Pi)}$ can be derived and hence risk measures, e.g. percentiles.

Conversely, the risk of systematic deviations, in particular due to uncertainty in future mortality trend, cannot be accounted for. To this purpose a more complex stochastic mortality model must be constructed.

3.4 Moving further to stochastic: allowing for uncertainty in future mortality trends

Experience on mortality suggests to adopt projected mortality tables (or laws) for the actuarial evaluation of life annuities (and other living benefits), i.e. to use mortality assumptions including a forecast of future mortality trends (see, for example, Tabeau et al. (2001), Pitacco et al. (2009) and references therein). Notwithstanding, whatever hypothesis is assumed the future trend is random, and hence an uncertainty risk arises. When this risk mainly refers to mortality improvements at old ages, it is usually called the "longevity risk".

Figure 3.3 illustrates two different modelling frameworks. When no randomness in future mortality trend is allowed for, and hence just one age-pattern of mortality is assumed (e.g. in terms of mortality intensity), a deterministic actuarial value of the life annuity follows. Conversely, if we recognize uncertainty in the future pattern of mortality, randomness of actuarial values emerges.

Uncertainty in mortality trend can be expressed and assessed in several ways. A very simple and practicable approach, basically consisting of two steps, is described below (for details see Olivieri (2001); an implementation was proposed by Olivieri and Pitacco (2003)).

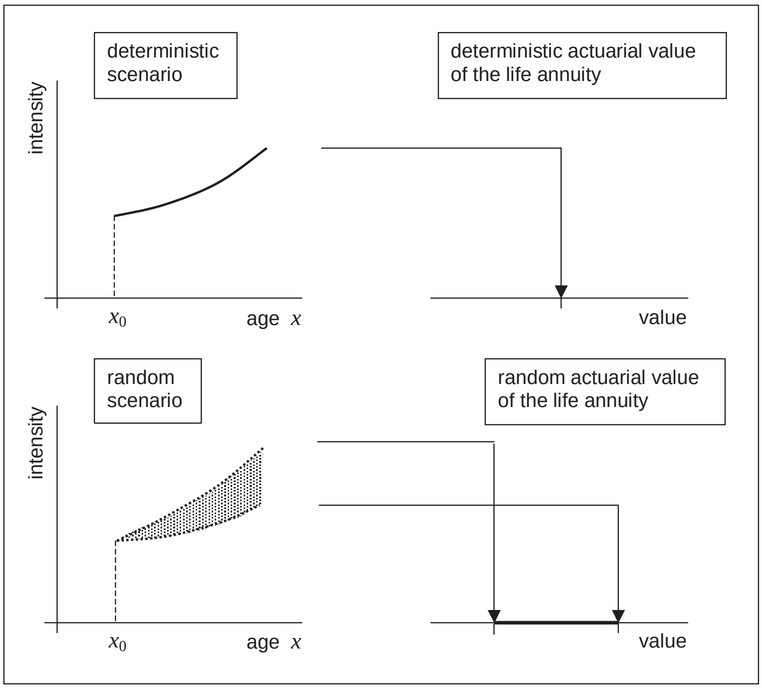

Figure 3.3: Mortality scenarios and actuarial values

1. Choose a set of projected mortality tables (or survival functions, or forces of mortality, etc.) in order to express several alternative hypotheses about future mortality evolution. So, it is possible to perform a scenario testing, assessing the range of variation of quantities such as cash flows, profits, portfolio reserves, and so on. This way, the sensitivity of these quantities to future mortality trend is investigated.

2. Assign non-negative normalized weights to the mortality hypotheses; the set of weights can be meant as a probability distribution on the space of hypotheses. Hence a stochastic approach can be adopted, calculating unconditional (i.e. non conditional on any particular hypothesis) variances, percentiles, etc., of the value of future cash flows, profits, and so on.

Let us express the uncertainty in future mortality trend assuming a (finite) set of hypotheses about the (projected) survival functions, namely a set of scenarios:

$$\mathcal{H} = \{H_1, H_2, \ldots, H_s\} \tag{3.29}$$

weighted with probabilities $\rho_1, \rho_2, \ldots, \rho_s$ respectively. The unknown scenario, out of the set $\mathcal{H}$, is denoted by $\tilde{H}$.

For a generic annuitant, the unconditional expected value and variance of future payments are:

$$\mathbb{E}[Y^{(j)}] = \mathbb{E}_\rho\Big[\mathbb{E}[Y^{(j)}|\tilde{H}]\Big] \tag{3.30}$$

$$\mathbb{V}\mathrm{ar}[Y^{(j)}] = \mathbb{E}_\rho\Big[\mathbb{V}\mathrm{ar}[Y^{(j)}|\tilde{H}]\Big] + \mathbb{V}\mathrm{ar}_\rho\Big[\mathbb{E}[Y^{(j)}|\tilde{H}]\Big] \tag{3.31}$$

The suffix ρ denotes that the expected value or the variance are calculated with respect to the mortality assumptions, weighted with the probabilities $\rho_1, \rho_2, \ldots, \rho_s$. Note that the first term of the variance takes account of random fluctuations around expected values, whilst the second one expresses systematic deviations of observed values from expected ones.

With reference to the portfolio, we have:

$$\mathbb{E}[Y^{(\Pi)}] = \mathbb{E}_\rho\Big[\mathbb{E}[Y^{(\Pi)}|\tilde{H}]\Big] = l_x\,\mathbb{E}_\rho\Big[\mathbb{E}[Y^{(j)}|\tilde{H}]\Big] = l_x\,\mathbb{E}[Y^{(j)}] \tag{3.32}$$

$$\mathbb{V}\mathrm{ar}[Y^{(\Pi)}] = \mathbb{E}_\rho\Big[\mathbb{V}\mathrm{ar}[Y^{(\Pi)}|\tilde{H}]\Big] + \mathbb{V}\mathrm{ar}_\rho\Big[\mathbb{E}[Y^{(\Pi)}|\tilde{H}]\Big] = l_x\,\mathbb{E}_\rho\Big[\mathbb{V}\mathrm{ar}[Y^{(j)}|\tilde{H}]\Big] + l_x^2\,\mathbb{V}\mathrm{ar}_\rho\Big[\mathbb{E}[Y^{(j)}|\tilde{H}]\Big] \tag{3.33}$$

As far as the risk index is concerned, we now obtain:

$$r(l_x) = \frac{\sqrt{\mathbb{V}\mathrm{ar}[Y^{(\Pi)}]}}{\mathbb{E}[Y^{(\Pi)}]} = \sqrt{\frac{1}{l_x}\,\frac{\mathbb{E}_\rho\Big[\mathbb{V}\mathrm{ar}[Y^{(j)}|\tilde{H}]\Big]}{\mathbb{E}^2[Y^{(j)}]} + \frac{\mathbb{V}\mathrm{ar}_\rho[\mathbb{E}\Big[Y^{(j)}|\tilde{H}]\Big]}{\mathbb{E}^2[Y^{(j)}]}} \tag{3.34}$$

The first term in the risk index shows that random fluctuations constitute a diversifiable risk, since (in relative terms) their effect can be reduced increasing the size of the portfolio. The second term shows, instead, that systematic deviations constitute a undiversifiable risk, viz the longevity risk, and hence unremovable by simply acting on the size of the portfolio. In particular, the asymptotic value

$$\lim_{l_x\to\infty} r(l_x) = \sqrt{\frac{\mathbb{V}\mathrm{ar}_\rho\Big[\mathbb{E}[Y^{(j)}|\tilde{H}]\Big]}{\mathbb{E}^2[Y^{(j)}]}} \tag{3.35}$$

can be taken as a measure of the unremovable part of the risk.

For further details and numerical examples, the reader can refer to Pitacco et al. (2009).

3.5 Risk mitigation

We now turn to RM techniques and, in particular, to portfolio strategies aiming at risk mitigation. Because of the complexity of the problem, we just refer to a portfolio of immediate life annuities, consisting in one cohort of annuitants. Nonetheless, an interesting framework can be built-up, within which a number of issues (insurance, financial, corporate issues, and so on) can be addressed under a unifying perspective.

A number of portfolio results can be taken as "metrics" to assess the effectiveness of portfolio strategies. In what follows, we focus on cash flows, which anyhow constitute the starting point from which other quantities are derived (say, annual profits). As we refer to single-premium annuities, cash flows are actually outflows; disregarding expenses, outflows are originated only by the payment of benefits.

In Fig. 3.4 a sequence of outflows is represented, together with a barrier (the "threshold") which represents a maintainable level of benefit payment. The threshold amount is financed first by (single) premiums via the portfolio mathematical reserve, and by shareholders' capital as the result of the allocation policy (consisting in specific capital allocations as well as accumulation of undistributed profits).

The situation occurring in Fig. 3.4, in which some annual cash flows are above the threshold level, should be clearly avoided. To lower the probability of such critical situations, the insurer can resort to various portfolio strategies, in the framework of the RM process.

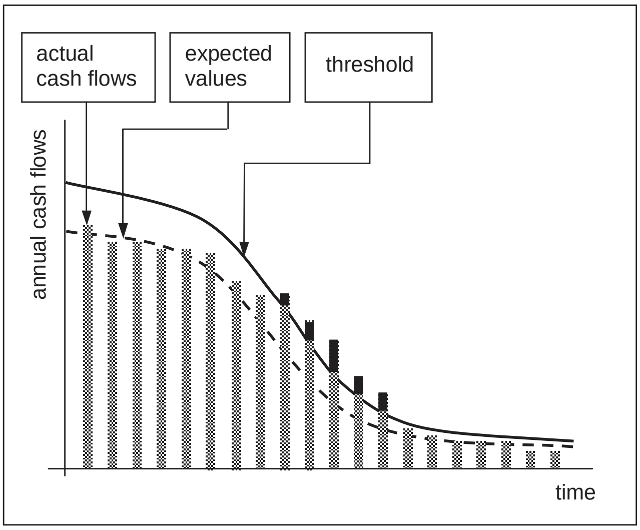

Figure 3.4: Annual cash flows in a life annuity portfolio (one cohort)

Figure 3.5 illustrates a wide range of portfolio strategies aiming at risk mitigation, meant as lowering the probability and the severity of events like the situation depicted in Fig. 3.4. In practical terms, a portfolio strategy can have as targets:

(a) an uplift of the maintainable annual cash flow, thus a higher threshold level;

(b) lower (and smoother) annual cash flows in the case of unanticipated improvements in portfolio mortality.

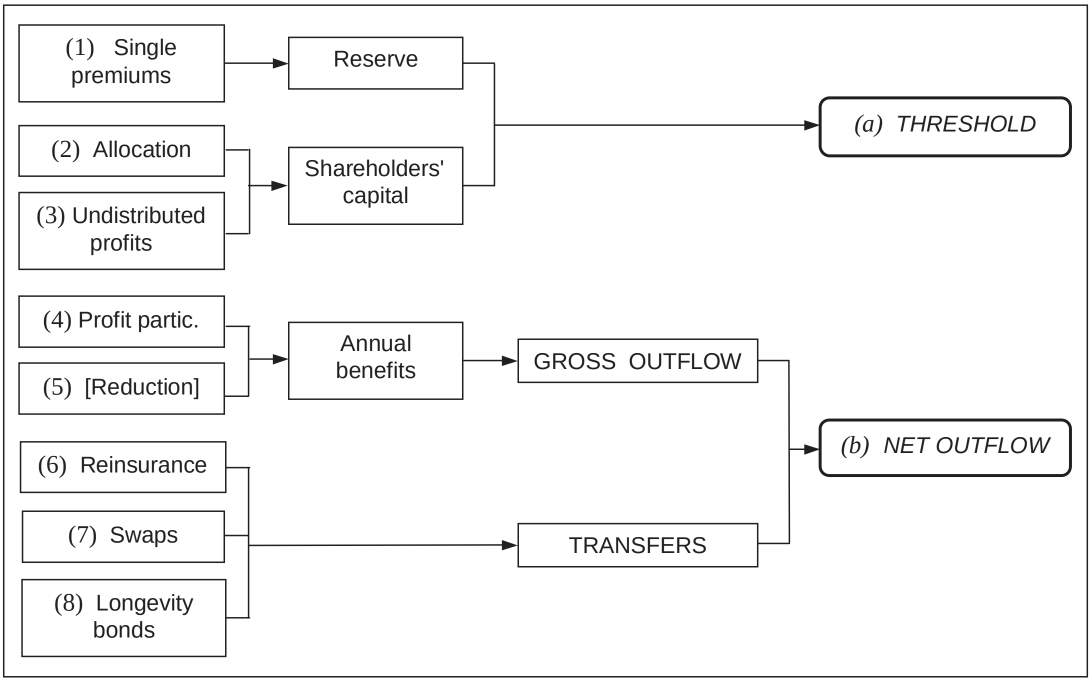

Figure 3.5: Portfolio strategies for risk mitigation

Both loss control and loss financing techniques (according to the RM language) can be adopted to achieve targets (a) and (b). Loss control techniques are mainly performed via product design, i.e. via an appropriate choice of the various items constituting an insurance product. In particular, loss prevention is usually meant as the RM technique aiming at mitigating the loss frequency, whereas loss reduction aims at lowering the severity of the possible losses.

The pricing of insurance products provides a tool for loss prevention. This portfolio strategy is represented by path $(1) \rightarrow (a)$ in Figure 3.5. Referring to a life annuity product, the following issues in particular should be taken into account.

– Mortality improvements require the use of a projected life table for pricing life annuities, as the hypothesis of "static" mortality should be rejected.

– Because of uncertainty in future mortality trend, a premium principle other than the traditional equivalence principle should be adopted. We note that, if the equivalence principle is adopted, the longevity risk can be accounted for only via a (rough) safety loading, constructed by increasing the survival probabilities resulting from the projected table. Actually, this approach is often adopted in actuarial practice.

– The presence, in an accumulation product such as an endowment insurance, of an option to annuitize at a fixed annuitization rate (the so-called Guaranteed Annuity Option) requires an accurate pricing model accounting for the value of the option itself.

In order to pursue loss reduction, it is necessary to control the amounts of benefits paid. Hence, some flexibility must be added to the annuity product. One action could be the reduction of the annuity amount as a consequence of an unanticipated mortality improvement (path $(5) \rightarrow (b)$ in Fig. 3.5); however, in this case the product would be a non-guaranteed annuity (though with a reasonable minimum amount guaranteed). A more practicable tool, consistent with the features of a guaranteed annuity, consists in reducing the level of investment profit participation while a poor mortality is experienced (path $(4) \rightarrow (b)$); it is worth stressing that undistributed profits also increase the shareholders' capital within the portfolio, hence uplifting the maintainable threshold (see $(3) \rightarrow (a)$).

Loss financing techniques require specific strategies involving the whole portfolio, and in some cases even other portfolios of the insurance company. Risk transfer can be realized via (traditional) reinsurance arrangements (see path $(6) \rightarrow (b)$), swap-like reinsurance ($(7) \rightarrow (b)$) and securitization, i.e. Alternative Risk Transfers (ART). ART require, when life annuities are concerned, the use of specific financial instruments, e.g. longevity bonds ($(8) \rightarrow (b)$) whose performance is linked to some measure of longevity in a given population.

The interest of reinsurance arrangements is mainly due to the possibility, for the reinsurer, of hedging the risk taken from the cedant via transfer to the capital market, namely via longevity bonds.

Some additional comments on risk transfers are worthwhile. Traditional reinsurance arrangements (e.g. surplus reinsurance, XL reinsurance, etc.) can be applied also to annuity portfolios, at least in principle (for example, see Olivieri (2005)). Anyhow, it should be stressed that risk transfer via traditional reinsurance mainly relies on the improved diversification of risks when these are taken by the reinsurer, thanks to a stronger pooling effect.

However, such an improvement can be achieved in relation to process risk (viz. mortality random fluctuations), whilst uncertainty risk (leading to systematic deviations) cannot be diversified "inside" the insurance-reinsurance process. Hence, to gain effectiveness reinsurance transfer must be completed with a further transfer, i.e. a transfer to capital markets. Such a transfer can be realized via bonds, whose yield is linked to some mortality/longevity index, so that the bonds themselves generate flows hedging the payment of benefits. Under a financial perspective, interest in such bonds clearly relies on a likely absence of correlation with other investment yields. For detailed information on this topic the reader can refer, for example, to Pitacco et al. (2009) and references therein.

To the extent that mortality/longevity risks are retained by an insurance company, the impact of a poor experience falls on the company itself. To meet an unexpected amount of obligations, an appropriate advance funding may provide a substantial help. To this purpose, shareholders' capital must be allocated to the annuity portfolio (see (2) $\rightarrow$ (a), as well as (3) $\rightarrow$ (a)), and the relevant amount should be determined aiming at insurer solvency. See, for example, Olivieri and Pitacco (2009). Conversely, the expression "no advance funding" (see Fig. 2.1) should be meant as no specific capital allocation facing the risks, whose impact will be (at least partially) met thanks to the capital required by legislation.

Hedging strategies in general consist in assuming a risk offsetting another risk borne by the insurer. In some cases, hedging strategies involve various portfolios or lines of business (LOBs), or even the whole insurance company, whence they cannot be placed in the portfolio framework as depicted by Figure 3.5.

In particular, a "natural" hedging consists in offsetting risks in different LOBs. For example, writing both life insurance providing death benefits and life annuities for similar groups of policyholders may help to provide a hedge against longevity risk. Such a hedge is usually named "across LOBs".

A natural hedge can be realised even inside an annuity portfolio, with the proviso that the product is no longer just a straight life annuity. Assume that the product consists in a life annuity combined with a death benefit with an amount decreasing as the age at death increases. Clearly, in the case of a mortality improvement higher than anticipated, death benefits lower than expected will be paid. Such a hedge is usually named "across time".

Clearly, mortality/longevity risks should be managed by the insurer via an appropriate mix of the tools described above. The choice of the RM tools is also driven by various interrelationships among the tools themselves. For example, the possibility of purchasing profitable reinsurance is strictly related with the features of the insurance products and, in particular, the life tables underlying the pricing as well as with the availability of ART for the reinsurer.

4 Deterministic versus stochastic modelling: a summary

Various modelling tools can be placed in a unifying context, which can provide significant value-added to life insurance teaching. In this section a brief summary is presented.

In order to deal with mortality/longevity risks, we have

(a) to choose an appropriate representation of some quantities directly related with mortality/survivorship, e.g. the number of insureds dying in the various years;

(b) to focus on portfolio results (e.g. cash flows, profits, etc.) which can significantly witness the financial impact of mortality/longevity risks.

While point (b) simply consists in an appropriate choice of one or more results and in expressing the relation between these and the quantities describing mortality, point (a) is a non-trivial issue of stochastic modelling. More precisely, a number of choices are actually available, ranging from a purely deterministic approach to very complex models allowing for uncertainty risk, as described in the previous sections.

Clearly stochastic mortality modelling can be placed in the (more general) framework of stochastic modelling for life insurance. In what follows we refer to this framework.

Assume that a result of interest, Y (say, a one-year cashflow), depends on some input variables, say X_1, X_2, X_3 (e.g. number of insureds alive, expenses, and so on)

$$Y = \Phi(X_1, X_2, X_3) \tag{4.1}$$

Figures 4.1 and 4.2 present various approaches to investigations about the result Y. Approach 1 is purely deterministic. Assigning specific values, x_1, x_2, x_3, to the three random variables, the corresponding outcome y of the result variable is simply calculated as $y = \Phi(x_1, x_2, x_3)$. First, it is interesting to note that classical actuarial calculations follow this approach, replacing random variables with their expected values, or anyhow with appropriate estimates. Secondly, in a more modern perspective this approach is adopted for example when performing stress testing (assigning to some variables "extreme" values), or in general scenario testing.

Randomness in input variables is, to some extent, accounted in iterative implementations of approach 1. Reasonable ranges for the outcomes of the input variables are chosen, and consequently a range $(y_{\min}, y_{\max})$ for the result Y can be derived.

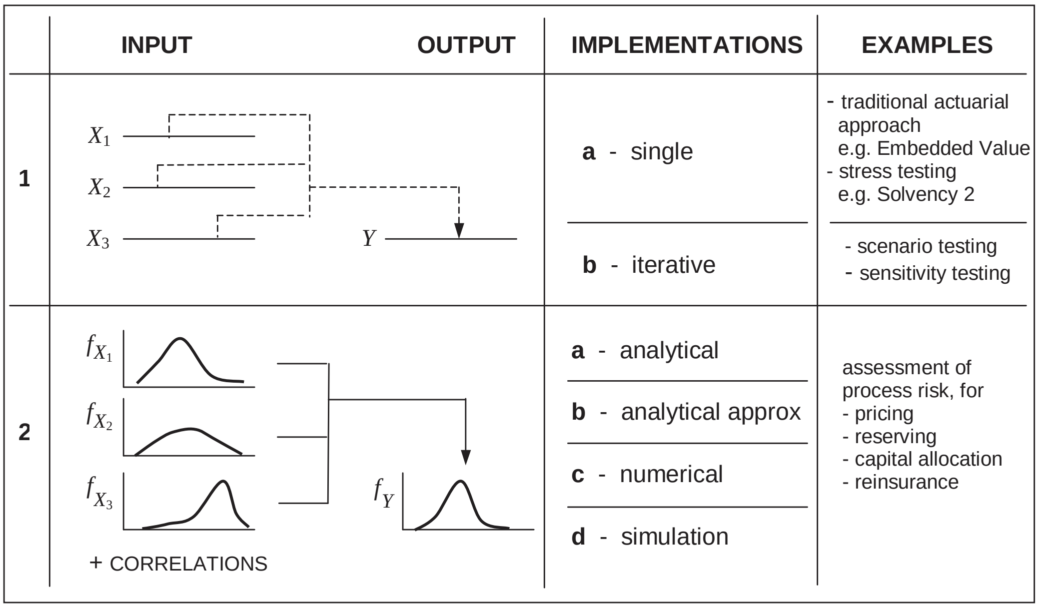

Figure 4.1: Modelling approaches (1)

Approach 2 provides a basic example of stochastic modelling, typically adopted for assessing the impact of process risk. A probabilistic structure is assigned to the input variables, in term of the joint probability distribution, or via marginal distributions (see

Fig. 4.1) and appropriate assumptions about correlations. The probability distribution of Y can be found using just analytical tools only in very simple (or simplified) circumstances. Numerical methods or stochastic simulation procedures help in most cases.

Dealing with uncertainty risk, in order to assess the impact of systematic deviations, is a crucial issue in particular in life insurance mathematics. Approach 3 simply consists in iterating the procedure implied by approach 2, each iteration corresponding to a specific assumption about the probability distribution of some input variables (the variable X_1 in Fig. 4.2), e.g. a specific set of values for the relevant parameters. Hence, a set of conditional distributions of the result Y is determined (see Sect. 3.4).

Finally, approach 4 aims at finding the unconditional probability distribution of the output variable Y, hence allowing for both process risk and uncertainty risk. A more complex probabilistic structure is then required, for example including a probability distribution over the set of assumptions (again, see Sect. 3.4).

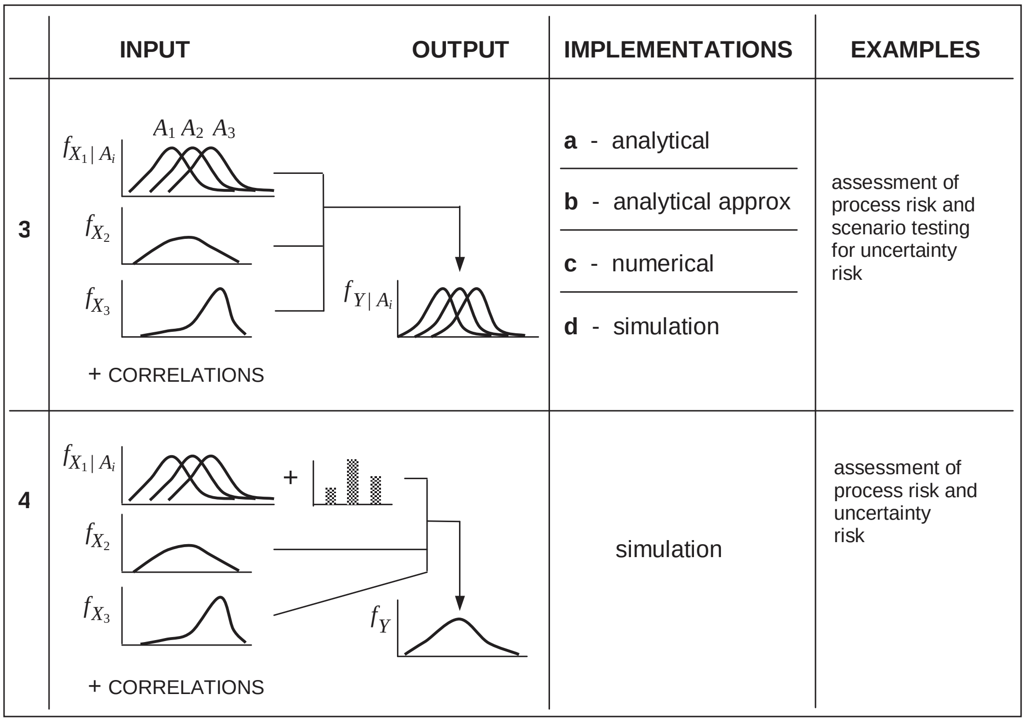

Figure 4.2: Modelling approaches (2)

5 Further examples

Mortality risks obviously impact on the performance of insurance portfolios providing benefits which only depend on insureds' lifetimes. However, a number of health insurance products (namely, still in the area of insurances of the person), especially those providing living benefits, are also affected by mortality risks (see for example Pitacco (2004c)). In particular:

- periodic premium arrangements lead to the accumulation of reserves whose sufficiency may be jeopardized by unanticipated mortality trends;

– huge fluctuations in mortality around its expected pattern may harm the technical equilibrium inside the portfolio.

Then, mortality risks (and, in particular, both the components process risk and uncertainty risk) should be carefully analyzed even when dealing with insurance products other than those providing benefits only depending on the insured's lifetime. In what follows, we briefly address three products in the area of health insurance.

5.1 Disability insurance (or Income Protection insurance)

Income Protection (IP) policies provide annuity benefits in the case of disability, due to accident or sickness. Appropriate modelling tools, for the calculation of premiums, reserves, expected profits, etc., are provided by Markov (and possibly semi-Markov) multistate models; see Haberman and Pitacco (1999), Wolthuis (2003), Pitacco (2004a). The probabilistic structure is usually assigned via transition intensities, from which transition probabilities are derived. Transition intensities quantify the demographical aspects of IP products, i.e. mortality of active and disabled people, disablement, recovery.

The traditional implementation of a multistate model leads to expected values only. Notwithstanding, randomness can be introduced into the model, in particular to assess the impact of random fluctuations in the numbers of transitions between states, namely deaths, disability inceptions, recoveries.

An interesting example of multistate implementation allowing for the process risk is provided by Haberman et al. (2004). The Authors consider the risk arising from the disability-claims process, and the risk originating from stochastic investment returns and interest rates. As regards the process risk inherent in the disability-claims process, the relevant impact on portfolio results is assessed via stochastic simulation, namely simulating the "paths" of the insureds through the states of the model.

It is worth noting that, from a teaching point of view, the proposed stochastic implementation offers a stimulating approach to multistate modelling, also providing suggestions for sensitivity analysis.

5.2 Post-retirement sickness covers

Uncertainty affecting whole-life sickness covers, in particular those providing post-retirement sickness benefits, originates from various causes. The following classification gives an insight into the randomness of a whole-life cover, and can help in appreciating the role of a premium system in contributing to the randomness itself. Uncertainty comes from:

1. the random number of claims in any given insured period;
2. the random amount (medical expenses refunded) relating to each claim;
3. the random duration of insured's life.

Note that items (1) and (2) are common to all covers in the framework of general insurance. The relevant effects can be faced by adopting appropriate premium calculation principles, and in particular by charging premiums with a suitable safety loading. However, it should be stressed that, in many insurance markets, paucity of data relating to very old ages increases the difficulty in assessments concerning (1) and (2).

Item (3) represents the mortality risk; its impact is strongly related to the premium arrangement adopted. Intuitively, the effect of this item vanishes when natural premiums are paid. Conversely, premium arrangements consisting in a single premium, e.g. paid on retirement, or in a temporary sequence of level premiums paid up to a fixed age imply the accumulation of reserves needed to finance the benefits due throughout the residual lifetime. Clearly, an unanticipated improvement in the mortality pattern may cause insufficiency of the reserves accumulated, then harming the technical equilibrium.

Thus, longevity risk may heavily affect whole-life sickness insurance products. In Olivieri and Pitacco (2002b) the impact of longevity risk on a portfolio of policies providing post-retirement sickness benefits is analyzed, and various premium systems are compared in order to assess their suitability when unexpected mortality improvements occur.

5.3 Long Term Care insurance

When sickness or disability benefits for the elderly are involved, attention should be devoted also to future morbidity trends. Olivieri and Pitacco (2002a), and Olivieri and Ferri (2003) have examined this problem for the case of Long Term Care (LTC) insurance products, which are particularly affected by senescent disability trends, in terms of both the probability of disablement and the length of the disability claim.

Appropriate tools for describing the structure of LTC covers are provided by multistate Markov (or semi-Markov) models. However, a high degree of uncertainty affects all transition intensities involved by the model, representing intensity of disablement (possibly split according to various degrees of severity), mortality of active people, and mortality of disabled people. So, uncertainty risk should be carefully taken into account.

In Olivieri and Ferri (2003) the risk profile of two LTC products (a “stand-alone” cover and the so-called “enhanced pension”) is analyzed, allowing for uncertainty in future transition intensities, and some solvency issues are addressed. A more RM oriented approach is adopted in Olivieri and Pitacco (2002a) where capital allocation policies and reinsurance arrangements are dealt with.

6 Concluding remarks

In recent decades teaching of life insurance mathematics has evolved thanks to important contributions from several scientific and technical areas. Some significant trends are listed below.

- Multistate Markov (and possibly semi-Markov) models have proved their power as a modelling tool for life insurance and other insurances of the person, since seminal contributions by J. Hoem (see Hoem (1969), Hoem (1988)). From a theoretical point of view, multistate models offer a unifying approach to the calculation of actuarial values. From a practical point of view, on one side they suggest rigorous settings for premium and reserve calculations, and on the other they provide the actuary with tools for checking the suitability of approximate calculation procedures.

- Numerical approaches and Montecarlo simulation methods have gained importance with the rapid progress in computer technology. The textbook by Daykin et al. (1994) shows a wide range of possibilities offered by simulation procedures when dealing with various causes of risk in both life and non-life insurance.

– Portfolio immunization, asset liability management, hedging opportunities, option pricing, etc. entered the framework of life insurance actuarial technique thanks to a number of contributions coming from Finance.

Nevertheless, actuarial mathematics and techniques still suffer a lack of contacts with firm management, corporate planning, etc., and this may result in actuarial teaching with contents rather far from the needs of insurance companies.

Valuable opportunities for starting an innovation process are currently offered by recent issues concerning solvency regulation. Actually, Solvency 2 places great emphasis on a RM approach to insurance business, involving all aspects of riskiness, mortality risks being of course included (see, for example, Sandström (2006)).

A final remark concerns teaching life insurance mathematics, meant as an appropriate mix of sound probabilistic principles and practical aspects of the insurance business. A rigorous probabilistic setting first emphasizes an "individual" approach to actuarial problems, centered on the random present value of benefits provided by insurance policies (see Sect. 3.3). However, when addressing the insurance company as an intermediary in the mutuality process originated by risk pooling, special attention should also be devoted to portfolio features. Hence, only looking at risks as a pool (see Sect. 3.3 again, and Sect. 3.4) can provide us with a logical framework suitable for dealing with riskiness according to a RM perspective.

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