Dependent Interest and Transition Rates in Life Insurance

Surrender modelling

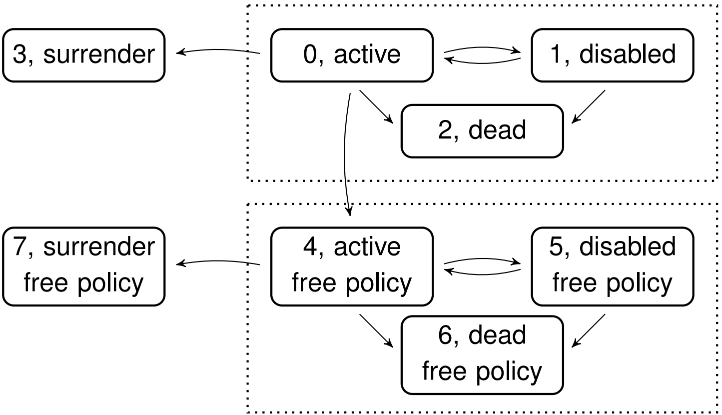
Kristian Buchardt
Ph.D.-student
http://math.ku.dk/~buchardt

PFA Pension

University of Copenhagen
Department of Mathematical Sciences

Example of a life insurance Markov model

Policyholder behaviour modelling



Introduction

- Examples of stochastic interest and/or transition rates
  - Stochastic mortality
  - Dependent interest and surrender(/policyholder behaviour)
  - Dependent mortality and disability?
- Stochastic transition rates: Also a way to model dependence between policyholders
  - Stochastic mortality affects all policyholders
  - Mass surrender:

    - large customers/brokers moving to another pension provider
- Stochastic transition rates necessary to model (Solvency II) capital requirement
- Hedging of interest rate risk affected by dependence:

  - Not enough to hedge expected cash flow

Affine processes: Mathematically tractable way of modelling stochastic interest and transition rates

Policyholder behaviour and Solvency II

Requirement to take policyholder behaviour into account.

- *Policyholders' option to surrender is often dependent on financial markets...*
- *In general, policyholders' behaviour should not be assumed to be independent of financial markets,...*

Quoted from "CEIOPS' Advice for Level 2 Implementing Measures on Solvency II: Technical provisions, Article 86 a, Actuarial and statistical methodologies to calculate the best estimate"

- Is interest and surrender rates dependent?
  - Well, if the market rate decrease to far below the guaranteed interest rate, people will probably keep their contract...

Continuous affine processes

$X(t) = (X_1(t), \dots, X_d(t))$ diffusion process,

$$\mathrm{d}X(t) = \left(b(t) + \sum_{i=1}^{d} \beta_i(t) X_i(t) \right) \mathrm{d}t + \sqrt{a(t) + \sum_{i=1}^{d} \alpha_i(t) X_i(t)}\, \mathrm{d}W(t)$$

$$X(0) = x$$

- Mathematically tractable class of processes
- Can ensure positivity

Theorems for affine processes

$k, l = 1, \ldots, d$ and $u, v \in [t, T]$,

$$\mathrm{E}\left[e^{-\int_t^T X_1(s)+\ldots+X_d(s)\,\mathrm{d}s} \middle| \mathcal{F}(t) \right] = e^{\phi(t,T)+\psi(t,T)^\top X(t)}$$

$$\mathrm{E}\left[e^{-\int_t^T X_1(s)+\ldots+X_d(s)\,\mathrm{d}s} X_k(u) \middle| \mathcal{F}(t) \right]$$

$$= e^{\phi(t,T)+\psi(t,T)^\top X(t)} \left(A^k(t,u) + B^k(t,u)^\top X(t) \right)$$

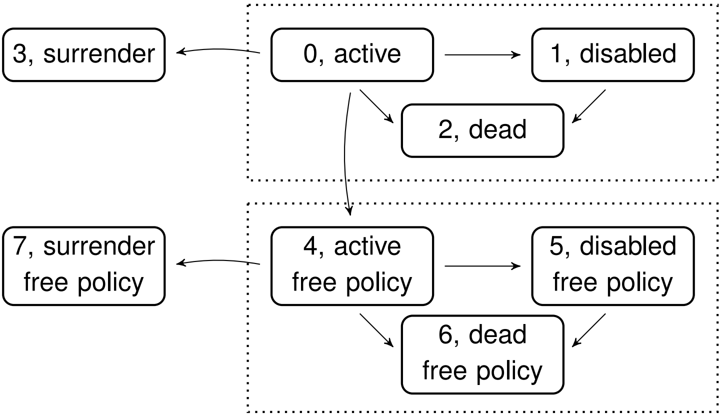
$$\mathrm{E}\left[e^{-\int_t^T X_1(s)+\ldots+X_d(s)\,\mathrm{d}s} X_k(u) X_l(v) \middle| \mathcal{F}(t) \right]$$

$$= e^{\phi(t,T)+\psi(t,T)^\top X(t)} \Big(C^{kl}(t,u,v) + D^{kl}(t,u,v)^\top X(t)$$

$$+ \left(A^k(t,u) + B^k(t,u)^\top X(t) \right) \left(A^l(t,v) + B^l(t,v)^\top X(t) \right) \Big),$$

$\phi, \psi, A^k, B^k, C^{kl}$ and D^{kl} solve ODEs.

Example of a life insurance Markov model



Hierarchical model:

$\Rightarrow$ Closed-form expressions for transition probabilities

$\Rightarrow$ Affine interest and transition rates are tractable

Simple savings product with surrender

Guaranteed interest rate r^*.

- If surrender does not occur: Payment 1 at T.
- If surrender at time s: Payment of $V^*(s) = e^{-r^*(T-s)}$.

Price at time 0, $V^*(0) = e^{-r^*T}$.

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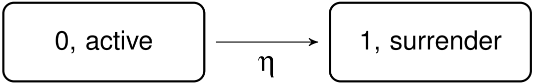


For notational simplicity: no insurance risk.
Market value at time t, for deterministic $r(t)$, $\eta(t)$,

$$
\begin{aligned}
V(t) &= \int_t^T e^{-\int_t^s (r(u)+\eta(u))\,\mathrm{d}u} \eta(s) V^*(s)\,\mathrm{d}s + e^{-\int_t^T (r(u)+\eta(u))\,\mathrm{d}u} \\
&= \int_t^T e^{-\int_t^s r(u)\,\mathrm{d}u} p_{00}(t,s) \eta(s) V^*(s)\,\mathrm{d}s + e^{-\int_t^T r(u)\,\mathrm{d}u} p_{00}(t,T)
\end{aligned}
$$

Stochastic interest and surrender rate model

Interest rate $r(t)$ and surrender rate $\eta(t)$ as dependent affine process,

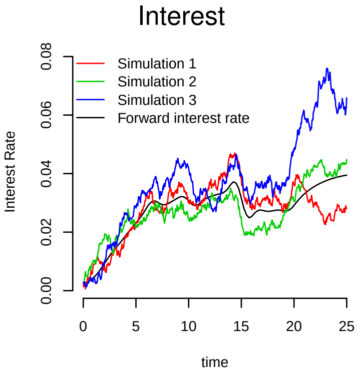
$$\begin{aligned} \mathrm{d}X_1(t) &= \beta(b_1(t) - X_1(t))\,\mathrm{d}t \\ &\qquad + \sigma_1 \left(\sqrt{1-\rho^2}\,\mathrm{d}W_1(t) + \rho\sqrt{X_2(t)}\,\mathrm{d}W_2(t) \right), \\ \mathrm{d}X_2(t) &= \beta(1 - X_2(t))\,\mathrm{d}t + \sigma_2\sqrt{X_2(t)}\,\mathrm{d}W_2(t), \\ X_2(0) &= 1, \end{aligned}$$

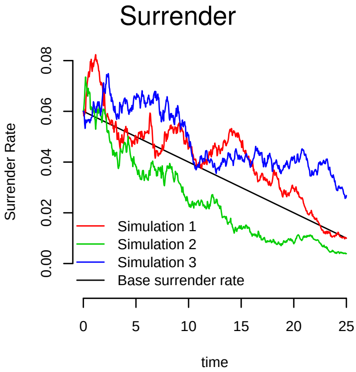
with $r(t) = X_1(t)$ and $\eta(t) = \eta^{\circ}(t)X_2(t)$.

- $(r(t), \eta(t))$ is a 2-dimensional affine process
- $\eta(t)$ non-negative
- $\mathrm{Corr}\,(r(t), \eta(t)) = \rho$

Stochastic interest and surrender rate model

- Parameters somewhat Solvency II (QIS 5) compatible
- Interest fitted to Danish FSA curve, 17. August 2012
- Correlation $\rho = 0.7$.





Market value with stochastic rates

Market value at time t. Expectation under suitable measure,

$$\begin{aligned} V(t) &= \mathrm{E}_t\left[\int_t^T e^{-\int_t^s (r(u)+\eta(u))\,\mathrm{d}u}\eta(s)V^*(s)\,\mathrm{d}s + e^{-\int_t^T (r(s)+\eta(s))\,\mathrm{d}s}\right] \\ &= \int_t^T e^{-\int_t^s (f_t^r(u)+f_t^\eta(u))\,\mathrm{d}u} f_t^\eta(s)V^*(s)\,\mathrm{d}s + e^{-\int_t^T (f_t^r(s)+f_t^\eta(s))\,\mathrm{d}s}. \end{aligned}$$

Generalised forward rates $f_t^r(s)$ and $f_t^\eta(s)$.

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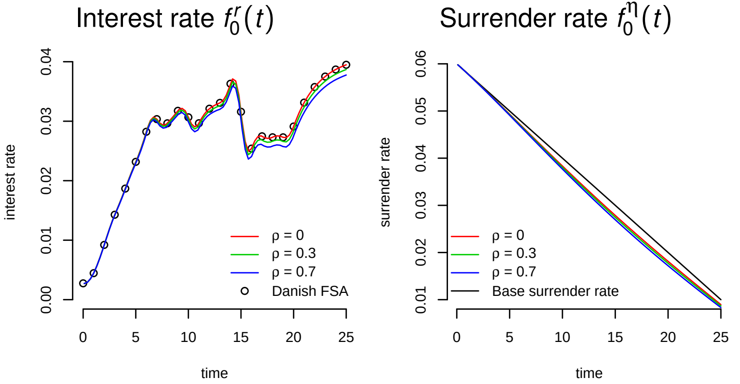
&= \int_t^T e^{-\int_t^s (f_t^r(u)+f_t^\eta(u))\,\mathrm{d}u} f_t^\eta(s)V^*(s)\,\mathrm{d}s + e^{-\int_t^T (f_t^r(s)+f_t^\eta(s))\,\mathrm{d}s}.
\end{aligned}
$$

Generalised forward rates $f_t^r(s)$ and $f_t^\eta(s)$.

$$
\begin{aligned}
\mathrm{E}_t\left[e^{-\int_t^T (r(s)+\eta(s))\,\mathrm{d}s}\right] &= e^{-\int_t^T (f_t^r(s)+f_t^\eta(s))\,\mathrm{d}s} \\
\mathrm{E}_t\left[e^{-\int_t^T (r(s)+\eta(s))\,\mathrm{d}s} r(s)\right] &= e^{-\int_t^T (f_t^r(s)+f_t^\eta(s))\,\mathrm{d}s} f_t^r(T) \\
\mathrm{E}_t\left[e^{-\int_t^T (r(s)+\eta(s))\,\mathrm{d}s} \eta(s)\right] &= e^{-\int_t^T (f_t^r(s)+f_t^\eta(s))\,\mathrm{d}s} f_t^\eta(T).
\end{aligned}
$$

Given by ODEs (A^k and B^k functions from the Theorems).

Generalised forward rates



- Generalised forward rates are decreasing in ρ

Market value

Market value, with surrender, $V(0)$,

r^*		4%	1%
	0	0.457	0.617
ρ	0.3	0.460	0.619
	0.7	0.463	0.622

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	0.7	0.463	0.622

Market value, no surrender ($\eta(t) = 0$),

$$V^{\mathrm{ns}}(0) = e^{-\int_0^{25} f_0^r(s)\,\mathrm{d}s} = 0.5037.$$

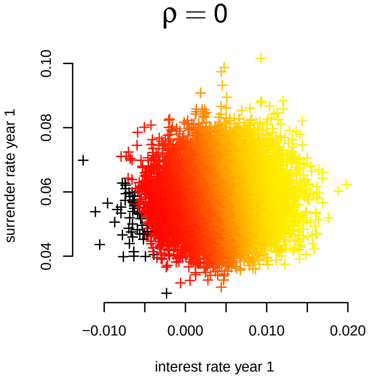
Account value $V^*(0) = e^{-25r^*}$,

4%	1%
0.3679	0.7788

Solvency II capital requirement: 4%

ρ	0.0	0.3	0.7
$\mathrm{E}\left[PV(0)\mid\mathcal{F}^{\mathbf{X}}(1)\right]-V(0)$	0.069	0.072	0.078

Figure: Interest and surrender rates in 1 year.
x/y axis: interest / surrender

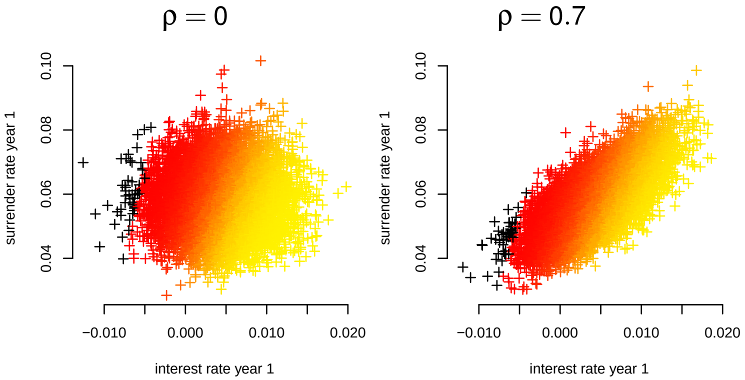


Solvency II capital requirement: 1%

ρ	0.0	0.3	0.7
$\mathrm{E}\left[PV(0)\middle|\mathcal{F}^{X}(1)\right] - V(0)$	0.077	0.072	0.060

Figure: Interest and surrender rates in 1 year.

x/y axis: interest / surrender



Mean-variance optimal static hedging

If interest and surrender are independent, hedge the expected cash flow:
For each year t, buy $c(t)$ bonds with expiry t,

$$c(t) = V^*(t)e^{-\int_0^t f_0^{\eta:\eta}(\tau)\,\mathrm{d}\tau} f_0^{\eta:\eta}(t)$$

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With dependence between interest and surrender,

$$c(t) = V^*(t) \left(\frac{e^{-\int_0^t f_0^{(2r+\eta):(2r+\eta)}(s)\,ds} f_0^{\eta:(2r+\eta)}(t)}{e^{-\int_0^t f_0^{2r:2r}(s)\,ds} - e^{-2\int_0^t f_0^{r:r}(s)\,ds}} - \frac{e^{-\int_0^t \left(f_0^{r:r}(s) + f_0^{(r+\eta):(r+\eta)}(s)\right) ds} f_0^{\eta:(r+\eta)}(t)}{e^{-\int_0^t f_0^{2r:2r}(s)\,ds} - e^{-2\int_0^t f_0^{r:r}(s)\,ds}} \right) + \dots$$

$\Rightarrow$ Optimal hedging of interest rate risk is dependence dependent.

Summary

- Affine processes allows for mathematically tractable models for stochastic and dependent interest and transition rates.
- In the example: Dependent interest and surrender increases market value.

  \- Policyholders behave more rational
- Solvency capital requirement can both be reduced and increased.
- Optimal interest rate hedging strategy affected by dependence structure.

References

- K. Buchardt (2012), *Continuous Affine Processes: Transformations, Markov Chains and Life Insurance.* (Preprint), Department of Mathematical Sciences, University of Copenhagen.
- K. Buchardt (2013), *Dependent interest and transition rates in life insurance.* (Preprint), Department of Mathematical Sciences, University of Copenhagen.

Thank you!