Dependent Interest and Transition Rates in Life Insurance

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Abstract

In order to find market consistent best estimates of life insurance liabilities modelled with a multi-state Markov chain, it is of importance to consider the interest and transition rates as stochastic processes, and hedging possibilities of the risks. This is usually done with an assumption of independence between the interest and transition rates. In this paper, it is shown how to valuate life insurance liabilities using affine processes for modelling dependent interest and transition rates. This approach leads to the introduction of so-called generalised forward rates. We propose a specific model for surrender modelling, and within this model the generalised forward rates are calculated, and the market value and the Solvency II capital requirement are examined for a simple savings contract.

Keywords: Affine Processes; Doubly Stochastic Process; Multi-state Life Insurance Models; Policyholder Behaviour; Solvency II

JEL Classification: G22

1 Introduction

Life insurance liabilities are traditionally modelled by a finite state Markov chain with deterministic interest and transition rates. In order to give a market consistent best estimate of the present value of future payments, it has become of increasing interest to let the interest and transition rates be modelled as stochastic processes. The stochastic modelling is important in order to consider hedging possibilities of the risks. With the Solvency II rules, stochastic modelling of the interest and transition rates is also important from a risk management perspective. Modelling the interest and transition

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rates as stochastic processes is traditionally done with an independence assumption. In this paper, we relax the independence assumption, and consider basic valuation with dependence between the interest and one or more transition rates. This is done with continuous affine processes for the modelling of the dependent rates. The study of valuation of life insurance liabilities with dependent rates leads to the definition of so-called generalised forward rates. These are natural quantities that appear in case of dependence, replacing the usual forward rates, which are not directly applicable. Using the theory of dependent affine rates, we consider the case of surrender modelling, and propose a specific model for dependent interest and surrender rates. This is of particular interest from a Solvency II point of view. Within this model, a simple savings contract with a buy-back option is considered. We calculate the generalised forward rates, the market value and the Solvency II capital requirement. This is done in part without hedging, and in part with a simple static hedging strategy. We then examine the effect of correlation between the interest and surrender rate.

The study of valuation of life insurance liabilities with stochastic interest and transition rates has received considerable attention during the last decades. Primarily the interest and mortality rates have been modelled as stochastic, which is often done with affine processes. For basic applications of affine processes for valuation of life insurance liabilities, see [1]. Possibilities of hedging can be considered, which is important for market consistent valuation, and for the study of valuation and hedging of life insurance liabilities with stochastic interest and mortality rates, see [7] and [6]. Another approach to modelling stochastic interest and mortality is taken in [15], where the interest and mortality is modelled within a finite state Markov chain setup. In this paper we extend the study of affine interest and transition rates to the case of dependence. We consider how to valuate life insurance liabilities when the interest and one or more transition rates are modelled as dependent affine processes. This is possible in any decrement/hierarchical Markov chain setup, that is, in Markov chains where, when the process leaves a state, it cannot return. We adopt the theory presented in [4], which is reviewed in Section 2 of this paper. This provides the foundation for the study of multidimensional affine processes in life insurance mathematics. The theory presented in [4] is based partly on a result in [8], and partly on general theory for multidimensional affine processes presented in [9].

In the financial literature, the concept of a forward interest rate exists, which is convenient for e.g. representing zero coupon bond prices. This quantity appears naturally in life insurance mathematics, when the interest rate is modelled as a stochastic process. If one also considers a stochastic mortality, independent of the interest rate, it becomes natural to define a forward mortality rate as well. With these forward rates, the expected present value of the life insurance liabilities looks particularly compelling. However, if one introduces dependence between the interest and mortality rates, the forward rates are no longer applicable. For a general discussion on forward rates, and their usefulness, see [14], wherein the case of dependence between the rates is discussed as well. In [11], alternative forward mortality rates are defined in order to handle the case of dependence. In this paper, we show that one of the forward mortality rates defined in [11] is in general not well defined. Instead, we introduce the concept of generalised forward rates, which appear naturally and can be used to express the expected present value of the life insurance liabilities in a convenient form. The generalised forward rates indeed generalise the usual definitions of forward rates, in the sense that when there is independence between the rates, the generalised forward rates equal the usual forward rates.

Modelling policyholder behaviour has become of increasing importance with the proposed Solvency II rules, where one is required to consider any dependence between the economic environment and policyholder behaviour, see Section 3.5 in [5]. The study of surrender behaviour can either be made using a rational approach, where the outset is, that the policyholders surrender the contract if it is rational from some economic viewpoint. This seems a bit extreme, given that this behaviour is not seen in practice. Another approach is the intensity approach, where the policyholders surrender randomly, regardless whether or not it is profitable in the current economic environment. This is not a perfect way of modelling either, since if the interest rates decrease a lot, a guarantee given in connection with the life insurance contract motivates the policyholders to keeping the contract. For an overview of some of the approaches, see [12]. In [10], an attempt is made on coupling the two approaches, using two different surrender rate models if it is rational or irrational, respectively, to surrender. In this paper, we propose another way of coupling the two approaches. We let the surrender rate be positively correlated with the interest rate, thus if the interest rate decreases a lot, the surrender rate also decreases, representing that the guarantee inherent in the life insurance contract is of value to the policyholder.

The Solvency II capital requirement is basically, that "the insurance company must have enough capital, such that the probability of default within the next year is less than 0.5%", representing that a default is a 200-year event. When the insurance company updates its mortality tables, or other transition rate tables, this represents a risk that must be taken into account when putting up the Solvency II capital requirement. Mathematically, this can be done using stochastic rates. For an examination of mortality modelling and the Solvency II capital requirement, see e.g. [2]. For a basic discussion of the mathematical formulation of the Solvency II capital requirement, see e.g. [3]. In this paper, we determine the Solvency II capital requirement for the simple savings contract, both in the case of no hedging strategy, and also in the case of a simple strategy where interest rate risk is hedged.

The structure of the paper is as follows. In Section 2, we present basic results on multidimensional continuous affine processes, which provides the foundation for the application of dependent affine processes in life insurance mathematics. In Section 3, we present the general life insurance setup with stochastic interest and transition rates, and in Section 4, we propose the definition of generalised forward rates, which is compared to the usual definition in Section 4.1. In Section 4.2, we discuss other definitions in the literature of forward rates in a dependent setup, and in particular, we show that the forward mortality rate for term insurances proposed in [11] does not always exist. In Section 5, we present a specific model for dependent interest and surrender rates. The model is introduced in Section 5.1. We first discuss how to find the Solvency II capital requirement, which is done in Section 5.3, and a simple hedging strategy for the interest rate risk is presented in Section 5.4. Numerical results are presented in Section 5.5, consisting of the generalised forward rates found, and the market value and Solvency II capital requirement, presented for different levels of correlation.

2 Continuous Affine Processes

The class of affine processes provides a method for modelling interest and transition rates, with the possibility of adding dependence. In this section, we consider general results about continuous affine processes, which we apply in this paper. For more details on the theory presented in this section, see [4].

Let $\mathbf{X}$ be a d-dimensional affine process, satisfying the stochastic differential equation

$$\mathrm{d}X(t) = (b(t) + \mathcal{B}(t)X(t))\,\mathrm{d}t + \rho(t, X(t))\,\mathrm{d}W(t),$$

where $\mathbf{W}$ is a d-dimensional Brownian motion. Here, $b : \mathbb{R}_+ \to \mathbb{R}^d$ is a vector function, and $\mathcal{B} : \mathbb{R}_+ \to \mathbb{R}^{d\times d}$ is a matrix function, where we denote column i by $\beta_i(t)$, so that $\mathcal{B}(t) = (\beta_1(t), \ldots, \beta_d(t))$. When squared, the volatility parameter function $\rho(t, x)$ must be affine in x, i.e.

$$\rho(t, x)\rho(t, x)^\top = a(t) + \sum_{i=1}^{d} \alpha_i(t)x_i,$$

for matrix functions $a : \mathbb{R}_+ \to \mathbb{R}^{d\times d}$ and $\alpha_i : \mathbb{R}_+ \to \mathbb{R}^{d\times d}$. Consider now affine transformations of $\mathbf{X}$, by defining a vector function $c : \mathbb{R}_+ \to \mathbb{R}^p$ and a matrix function $\Gamma : \mathbb{R}_+ \to \mathbb{R}^{p\times d}$, thereby defining the p-dimensional process,

$$Y(t) = c(t) + \Gamma(t)X(t). \tag{2.1}$$

We think of $\mathbf{X}$ as socio-economic driving factors, and then $\mathbf{Y}$ is a collection of the stochastic interest rate and/or transition rates. In this section, we work in a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with the filtration $\mathbb{F} = (\mathcal{F}(t))_{t\in\mathbb{R}_+}$ generated by the Brownian motion $\mathbf{W}$.

For applications of $\mathbf{Y}$ as interest and as transition rates in finite state Markov chain models, we present some essential relations. The results hold under certain regularity conditions, for details see [4]. Denote by $\mathbf{1}$ a vector with 1 in all entries, where the dimension is implicit. Also, denote by $\gamma_i(t)$ the sum of the ith column in $\Gamma(t)$, i.e. $\gamma_i(t) = \mathbb{1}^\top \Gamma(t) e_i$, where e_i is the ith unit vector, $i = 1, \dots, d$.

The first relation, the basic pricing formula, is for $0 \leq t \leq T$ given by

$$\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\,\mathrm{d}s} \middle| \mathcal{F}(t) \right] = e^{\phi(t,T) + \psi(t,T)^\top X(t)}, \tag{2.2}$$

where $\phi(t,T)$ is a real function, and $\psi(t,T)$ is a p-dimensional function, given by the system of differential equations,

$$\begin{aligned} \frac{\partial}{\partial t}\phi(t,T) &= -\frac{1}{2}\psi(t,T)^\top a(t)\psi(t,T) - b(t)^\top \psi(t,T) + \mathbb{1}^\top c(t), \\ \frac{\partial}{\partial t}\psi_i(t,T) &= -\frac{1}{2}\psi(t,T)^\top \alpha_i(t)\psi(t,T) - \beta_i(t)^\top \psi(t,T) + \gamma_i(t), \quad i = 1, \dots, d, \end{aligned} \tag{2.3}$$

with boundary conditions $\phi(T,T) = 0$ and $\psi(T,T) = 0$.

For the second relation, let a vector $\kappa \in \mathbb{R}^p$ be given, and let $u \in [t,T]$ be some time point. Then,

$$\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\,\mathrm{d}s} \kappa^\top Y(u) \middle| \mathcal{F}(t) \right] = e^{\phi(t,T) + \psi(t,T)^\top X(t)} \Big(A(t,T,u) + B(t,T,u)^\top X(t) \Big), \tag{2.4}$$

where (ϕ, ψ) is given by (2.3) as above, A is a real function and B is a vector function, given by the system of differential equations,

$$\begin{aligned} \frac{\partial}{\partial t}A(t,T,u) &= -\psi(t,T)^\top a(t)B(t,T,u) - b(t)^\top B(t,T,u), \\ \frac{\partial}{\partial t}B_i(t,T,u) &= -\psi(t,T)^\top \alpha_i(t)B(t,T,u) - \beta_i(t)^\top B(t,T,u), \quad i = 1, \dots, d, \end{aligned} \tag{2.5}$$

with boundary conditions $A(u,T,u) = \kappa^\top c(u)$ and $B(u,T,u) = \kappa^\top \Gamma(u)$. A particular example of importance is $\kappa = e_k$ for some $k = 1, \dots, p$, and in this case, we write A^k and B^k to emphasize the dependence on k. This second relation (2.4) is proven in [8] for $u = T$, and the extension to the case $u < T$ is for example given in [4].

The third relation is, for another time point $v \in [t,T]$, and two integers $k, l = 1, \dots, p$, given by

$$\begin{aligned}\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\,\mathrm{d}s} Y_k(u) Y_l(v) \,\middle|\, \mathcal{F}(t)\right] &= e^{\phi(t,T)+\psi(t,T)^\top X(t)} \\ \times \Big\{ \Big(A^k(t,T,u) + B^k(t,T,u)^\top X(t)\Big) &\Big(A^l(t,T,v) + B^l(t,T,v)^\top X(t)\Big) \\ + C^{kl}(t,T,u,v) + D^{kl}(t,T,u,v)^\top X(t)\Big\}&,\end{aligned} \tag{2.6}$$

where (ϕ, ψ) solves (2.3) and (A^k, B^k) and (A^l, B^l) both solve (2.5) with boundary conditions $A^k(u,T,u) = e_k^\top c(u)$, $B^k(u,T,u) = e_k^\top \Gamma(u)$ and $A^l(v,T,v) = e_l^\top c(v)$, $B^l(v,T,v) = e_l^\top \Gamma(v)$, respectively. The functions C^{kl} and D^{kl} are determined by the following system of differential equations,

$$\begin{aligned}\frac{\partial}{\partial t} C^{kl}(t,T,u,v) &= -B^k(t,T,u)^\top a(t) B^l(t,T,v) \\ &\quad - \psi(t,T)^\top a(t) D^{kl}(t,T,u,v) - b(t)^\top D^{kl}(t,T,u,v), \\ \frac{\partial}{\partial t} D_i^{kl}(t,T,u,v) &= -B^k(t,T,u)^\top \alpha_i(t) B^l(t,T,v) \\ &\quad - \psi(t,T)^\top \alpha_i(t) D^{kl}(t,T,u,v) - \beta_i(t)^\top D^{kl}(t,T,u,v),\end{aligned} \tag{2.7}$$

for $i = 1, \ldots, d$, with boundary conditions[1] $C^{kl}(u \wedge v, T, u, v) = 0$ and $D^{kl}(u \wedge v, T, u, v) = 0$. This result is proven in [4].

3 The Life Insurance Model

Consider the usual life insurance setup. Let $\mathbf{Z} = (Z(t))_{t \in \mathbb{R}_+}$ be a Markov process in the finite state space $\mathcal{J}$, indicating the state of the insured. The distribution of $\mathbf{Z}$ is defined via the transition rates $(\mu_{ij}(t))_{t \in \mathbb{R}_+}$, $i, j \in \mathcal{J}$. With $(N_{ij}(t))_{t \in \mathbb{R}_+}$, $i, j \in \mathcal{J}$ being the process that counts the number of jumps for $\mathbf{Z}$ from state i to j, the compensated process

$$N_{ij}(t) - \int_0^t 1_{(Z(s-)=i)} \mu_{ij}(s)\,\mathrm{d}s$$

is a martingale. We can allow the transition rates (μ_{ij}) to be stochastic. In this case, the distribution of $\mathbf{Z}$ is defined conditionally on the transition rates.

We model the transition rates as a time-dependent affine transformation of a d-dimensional continuous affine process $\mathbf{X}$. That is, for functions $c : \mathbb{R}_+ \to \mathbb{R}^p$ and $\Gamma : \mathbb{R}_+ \to \mathbb{R}^{p \times d}$, let $\mathbf{Y}$ be defined as

$$Y(t) = c(t) + \Gamma(t) X(t).$$

[1] The notation $x \wedge y = \min\{x, y\}$ is used.

Hence, each of the stochastic transition rates are modelled as an element in $\mathbf{Y}$.

The interest rate process $(r(t))_{t\in\mathbb{R}_+}$ is also allowed to be stochastic. This is modelled in the same way, by specifying r as an element in $\mathbf{Y}$. By the design of Γ and $\mathbf{X}$, the interest and transition rates can be dependent, independent or deterministic.

Let the filtrations $\mathbb{F}^{\mathbf{Z}} = (\mathcal{F}^{\mathbf{Z}}(t))_{t\in\mathbb{R}_+}$ and $\mathbb{F}^{\mathbf{X}} = (\mathcal{F}^{\mathbf{X}}(t))_{t\in\mathbb{R}_+}$ be the ones generated by the processes $\mathbf{Z}$ and $\mathbf{X}$, respectively, satisfying the usual hypothesis. We consider the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F} = (\mathcal{F}(t))_{t\in\mathbb{R}_+}$ is given by $\mathcal{F}(t) = \mathcal{F}^{\mathbf{Z}}(t) \vee \mathcal{F}^{\mathbf{X}}(t)$.

We consider a life insurance policy, with payments specified by the process $\mathbf{B} = (B(t))_{t\in\mathbb{R}_+}$, such that $B(t)$ is the total payments until time t. Then we can think of $\mathrm{d}B(t)$ as the payment at time t, and we can specify $\mathbf{B}$ as

$$\mathrm{d}B(t) = \sum_{i\in\mathcal{J}} 1_{(Z(t)=i)} b_i(t)\,\mathrm{d}t + \sum_{\substack{i,j\in\mathcal{J}\\ i\neq j}} b_{ij}(t)\,\mathrm{d}N_{ij}(t),$$

for deterministic payment functions b_i and b_{ij}, $i, j \in \mathcal{J}$. Then $b_i(t)$ is the payment while in state i at time t, and $b_{ij}(t)$ is the payment if jumping from state i to j at time t.

The present value at time t of the future payments associated with the life insurance policy is given by

$$PV(t) = \int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}B(s).$$

For reserving and pricing, one considers the expected present value

$$V(t) = \mathrm{E}\left[\int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}B(s)\middle|\,\mathcal{F}(t)\right],$$

where the expectation is taken using a market, risk neutral or pricing measure. For actually calculating $V(t)$, the tower property is applied, that is, we condition on $\mathcal{F}^{\mathbf{X}}(\infty)$ to get

$$V^{\mathbf{X}}(t) = \mathrm{E}\left[\int_t^\infty e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}B(s)\middle|\,\mathcal{F}^{\mathbf{Z}}(t) \vee \mathcal{F}^{\mathbf{X}}(\infty)\right],$$

so that $V(t) = \mathrm{E}\left[V^{\mathbf{X}}(t)\middle|\,\mathcal{F}(t)\right]$. Here, $V^{\mathbf{X}}(t)$ is the reserve conditional on the interest and transition rates, thus corresponding to the case of deterministic rates. When valuating $V^{\mathbf{X}}$ we need the conditional distribution of $\mathbf{Z}$, and thus $\mathbf{B}$, given the transition rates. By construction this is known, and well-established theory about life insurance reserves with deterministic interest and transition rates (see e.g. [13]) hold.

Example 3.1. Consider a surrender model with 3 states $\mathcal{J} = \{0, 1, 2\}$, corresponding to *alive*, *dead* and *surrendered* respectively. The Markov model is shown in Figure 1. Let the transition rate from state *alive* to state *dead*, i.e. the mortality rate, be deterministic. We model the interest rate r and the surrender rate η as dependent affine processes in the form,

$$(r(t), \eta(t))^\top = c(t) + \Gamma(t)X(t),$$

for a d-dimensional affine process $\mathbf{X}$. Hence, this specification is analog to (2.1). By the design of $\mathbf{X}$, the processes $\mathbf{X}_i$, $i = 1, \ldots, d$ can be dependent processes, such that the interest rate r and the surrender rate η can be dependent processes.

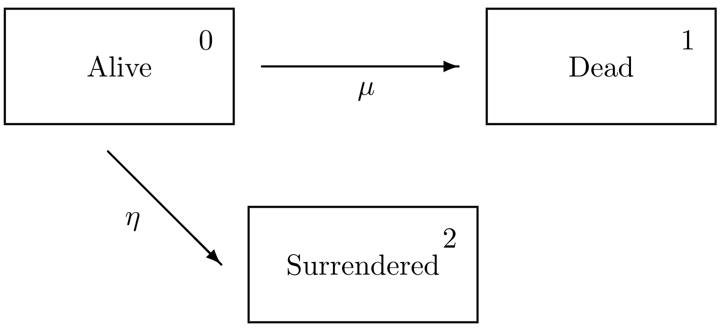

Figure 1: *Markov model for the survival-surrender model.*

Let the payments be defined by

$$\mathrm{d}B(t) = b(t)1_{(Z(t)=0)}\,\mathrm{d}t + b^d(t)\,\mathrm{d}N_{01}(t) + U(t)\,\mathrm{d}N_{02}(t),$$

where $b(t)$ is the continuous payment rate at time t while alive, $b^d(t)$ is the single payment if death occurs at time t, and $U(t)$ is the payment upon surrender at time t. The payment functions are deterministic.

Conditioning on the intensitites, the expected present value $V^{\mathbf{X}}(t)$ is the classic result,

$$\begin{aligned} V^{\mathbf{X}}(t) &= \mathrm{E}\left[PV(t) \mid \mathcal{F}^{\mathbf{X}}(t), Z(t) = 0\right] \\ &= \int_t^\infty e^{-\int_t^s (r(\tau)+\mu(\tau)+\eta(\tau))\,\mathrm{d}\tau} \left(b(s) + \mu(s)b_d(s) + \eta(s)U(s)\right)\,\mathrm{d}s, \end{aligned}$$

see e.g. [13]. Removing the condition, we find, using Theorem (2.2) and (2.4),

$$V(t) = \mathrm{E}\left[V^{\mathbf{X}}(t)\middle|\, \mathcal{F}(t)\right]$$

$$
\begin{aligned}
&= \int_t^\infty e^{-\int_t^s \mu(\tau)\,\mathrm{d}\tau} \bigg\{ \mathrm{E}\left[e^{-\int_t^s (r(\tau)+\eta(\tau))\,\mathrm{d}\tau} \middle| \mathcal{F}(t) \right] (b(s) + \mu(s) b_d(s)) \\
&\quad + \mathrm{E}\left[e^{-\int_t^s (r(\tau)+\eta(\tau))\,\mathrm{d}\tau} \eta(s) \middle| \mathcal{F}(t) \right] U(s) \bigg\} \,\mathrm{d}s \\
&= \int_t^\infty e^{-\int_t^s \mu(\tau)\,\mathrm{d}\tau + \phi(t,s) + \psi(t,s)^\top X(t)} \Big(b(s) + \mu(s) b_d(s) \\
&\quad + \Big(A^\eta(t,s,s) + B^\eta(t,s,s)^\top X(t) \Big) U(s) \Big) \,\mathrm{d}s.
\end{aligned}
$$

○

4 Generalised Forward Rates

The form of $V(t)$ in Example 3.1 motivates the definition of quantities similar to forward rates, that can be used to express the solution. In particular, this leads to a forward interest rate, but this is in general not equal the forward rate obtained using the usual definition. Hence, we apply the term *generalised forward rates*.

Let $\mathbf{X}$, $c(t)$ and $\Gamma(t)$ be given, and let $\mathbf{Y}$ be of the form (2.1). Consider first the motivating calculations,

$$
\begin{aligned}
&\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\,\mathrm{d}s} \mathbb{1}^\top Y(T) \middle| \mathcal{F}(t) \right] \\
&= -\frac{\partial}{\partial T} \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\,\mathrm{d}s} \middle| \mathcal{F}(t) \right] \\
&= -\frac{\partial}{\partial T} e^{\phi(t,T) + \psi(t,T)^\top X(t)} \\
&= e^{\phi(t,T) + \psi(t,T)^\top X(t)} \left(-\frac{\partial}{\partial T} \phi(t,T) + X(t)^\top \left(-\frac{\partial}{\partial T} \psi(t,T) \right) \right),
\end{aligned}
$$

where we interchanged integration and differentiation, and applied (2.2). On the other hand, if we instead apply (2.4) with $\kappa = \mathbb{1}$, we find

$$
\begin{aligned}
&\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\,\mathrm{d}s} \mathbb{1}^\top Y(T) \middle| \mathcal{F}(t) \right] \\
&= e^{\phi(t,T) + \psi(t,T)^\top X(t)} \Big(A(t,T,T) + X(t)^\top B(t,T,T) \Big), \\
&= e^{\phi(t,T) + \psi(t,T)^\top X(t)} \left(\sum_{k=1}^p A^k(t,T,T) + X(t)^\top \sum_{k=1}^p B^k(t,T,T) \right),
\end{aligned}
$$

where (A^k, B^k), $k = 1, \ldots, p$ are solutions to (2.5) with boundary conditions given by $\kappa = e_k$, i.e. $A^k(T,T,T) = e_k^\top c(T)$ and $B^k(T,T,T) = e_k^\top \Gamma(T)$. The last equality sign is obtained using the relations $\sum_{k=1}^p A^k(t,T,T) = A(t,T,T)$ and $\sum_{k=1}^p B^k(t,T,T) = B(t,T,T)$, which hold since (A, B) also solves the linear system of differential equations

(2.5), with boundary conditions given by $\kappa = \mathbb{1}$. Gathering the two calculations above, we conclude that

$$-\frac{\partial}{\partial T}\phi(t,T) = \sum_{k=1}^{p} A^k(t,T,T), \qquad -\frac{\partial}{\partial T}\psi(t,T) = \sum_{k=1}^{p} B^k(t,T,T),$$

and, in particular, since $\phi(t,t) = 0$ and $\psi(t,t) = 0$, that

$$\phi(t,T) = -\int_t^T \sum_{k=1}^{d} A^k(t,s,s)\,\mathrm{d}s, \qquad \psi(t,T) = -\int_t^T \sum_{k=1}^{d} B^k(t,s,s)\,\mathrm{d}s. \tag{4.1}$$

Definition 4.1. *Let* $\mathbf{X}$ *be a* d*-dimensional continuous affine process, and let* c *and* Γ *be given, such that* $\mathbf{Y}$ *from* (2.1) *is defined. Let* $s \le t$ *and* $k = \{1,\dots,d\}$*. The* generalised forward rate $f_t^k(s)$ *for the stochastic rate* $Y_k(s)$ *at time* t *is then given by*

$$f_t^k(s) = A^k(t,s,s) + X(t)^\top B^k(t,s,s),$$

where (A^k, B^k) *solves the system of differential equations* (2.5)*, with boundary conditions given by* $\kappa = e_k$*.*

Remark 4.2. Using the notation of the generalised forward rates, we can then express the relation (2.3), and for $u = T$ also the relation (2.5), as

$$\begin{aligned} \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\,\mathrm{d}s} \,\middle|\, \mathcal{F}(t)\right] &= e^{-\int_t^T \sum_{i=1}^d f_t^i(s)\,\mathrm{d}s}, \\ \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\,\mathrm{d}s} Y_k(T) \,\middle|\, \mathcal{F}(t)\right] &= e^{-\int_t^T \sum_{i=1}^d f_t^i(s)\,\mathrm{d}s} f_t^k(T). \end{aligned} \tag{4.2}$$

$\diamond$

Example 4.3. (Example 3.1 continued) Using the definition of the generalised forward rates, we can write the expected present value as,

$$V(t) = \int_t^\infty e^{-\int_t^s \left(f_t^r(\tau) + \mu(\tau) + f_t^\eta(\tau)\right)\mathrm{d}\tau} \Big(b(s) + \mu(s) b_d(s) + f_t^\eta(s) U(s) \Big)\,\mathrm{d}s.$$

We see that the expected present value is of the same form as the formula that appears in the case of deterministic rates, but with the interest and surrender rates exchanged by the corresponding generalised forward rates. Note that we used a slightly different notation, such that we write f^r instead of f^1 and f^η instead of f^2.

Often we want to consider both the quantity

$$\mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y(s)\,\mathrm{d}s} \,\middle|\, \mathcal{F}(t)\right] = \mathrm{E}\left[e^{-\int_t^T (r(s)+\eta(s))\,\mathrm{d}s} \,\middle|\, \mathcal{F}(t)\right],$$

where $Y(t) = (r(t), \eta(t))$, as well as the quantities arising from the models $Y^1(t) = (r(t), 0)$ and $Y^2(t) = (0, \eta(t))$,

$$\begin{aligned} \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y^1(s)\,\mathrm{d}s} \middle| \mathcal{F}(t) \right] &= \mathrm{E}\left[e^{-\int_t^T r(s)\,\mathrm{d}s} \middle| \mathcal{F}(t) \right], \\ \mathrm{E}\left[e^{-\int_t^T \mathbb{1}^\top Y^2(s)\,\mathrm{d}s} \middle| \mathcal{F}(t) \right] &= \mathrm{E}\left[e^{-\int_t^T \eta(s)\,\mathrm{d}s} \middle| \mathcal{F}(t) \right]. \end{aligned}$$

In such cases, we add a more detailed superscript to the forward rates f, and specify the model we think of after a colon. That is, we write

$$\mathrm{E}\left[e^{-\int_t^T (r(s)+\eta(s))\,\mathrm{d}s} \middle| \mathcal{F}(t) \right] = e^{-\int_t^T (f_t^{r:(r+\eta)}(s) + f_t^{\eta:(r+\eta)}(s))\,\mathrm{d}s},$$

as well as

$$\begin{aligned} \mathrm{E}\left[e^{-\int_t^T r(s)\,\mathrm{d}s} \middle| \mathcal{F}(t) \right] &= e^{-\int_t^T f_t^{r:r}(s)\,\mathrm{d}s}, \\ \mathrm{E}\left[e^{-\int_t^T \eta(s)\,\mathrm{d}s} \middle| \mathcal{F}(t) \right] &= e^{-\int_t^T f_t^{\eta:\eta}(s)\,\mathrm{d}s}. \end{aligned}$$

Note that $f_t^{r:r}(s)$ and $f_t^{\mu:\mu}(s)$ are the usual forward rates. ○

4.1 Comparison With The Usual Forward Interest Rate

Let the model $Y(t) = c(t) + \Gamma(t)X(t)$ be given, for $d > 1$, and let $r(t) = Y_1(t)$ be the interest rate. The *forward interest rate* is the function $g_t(s)$ that satisfies

$$\mathrm{E}\left[e^{-\int_t^T r(s)\,\mathrm{d}s} \middle| \mathcal{F}(t) \right] = e^{-\int_t^T g_t(s)\,\mathrm{d}s}.$$

This function also satisfies, as can be shown by differentiation,

$$\mathrm{E}\left[e^{-\int_t^T r(s)\,\mathrm{d}s} r(T) \middle| \mathcal{F}(t) \right] = e^{-\int_t^T g_t(s)\,\mathrm{d}s} g_t(T). \tag{4.3}$$

The generalised forward rate for the interest rate in our model $\mathbf{Y}$, as defined in Definition 4.1, is denoted $f_t^r(s)$. It satisfies,

$$\mathrm{E}\left[e^{-\int_t^T (r(s)+Y_2(s)+\ldots+Y_d(s))\,\mathrm{d}s} r(T) \middle| \mathcal{F}(t) \right] = e^{-\int_t^T \left(f_t^r(s)+f_t^2(s)+\ldots+f_t^d(s)\right)\mathrm{d}s} f_t^r(T), \tag{4.4}$$

where the other forward rates $f_t^i(s)$ satisfies analogue relations. We note, that while the usual forward interest rate is defined for any stochastic interest rate, the generalised forward rates from Definition 4.1 are only defined for (continuous) affine stochastic rates.

In the case that $\mathbf{r} = (r(t))_{t\in\mathbb{R}_+}$ is independent of $\mathbf{Y}_2, \ldots, \mathbf{Y}_d$, the generalised forward rate for the interest $\mathbf{r}$ simplifies to the usual forward interest rate. This can be seen by two simple calculations. First, see that

$$\begin{aligned} & e^{-\int_t^T \left(f_t^r(s)+f_t^2(s)+\ldots+f_t^d(s)\right) \mathrm{d}s} \\ & = \mathrm{E}\left[e^{-\int_t^T (r(s)+Y_2(s)+\ldots+Y_d(s))\,\mathrm{d}s} \middle| \mathcal{F}(t)\right] \\ & = \mathrm{E}\left[e^{-\int_t^T r(s)\,\mathrm{d}s} \middle| \mathcal{F}(t)\right] \mathrm{E}\left[e^{-\int_t^T (Y_2(s)+\ldots+Y_d(s))\,\mathrm{d}s} \middle| \mathcal{F}(t)\right] \\ & = e^{-\int_t^T g_t(s)\,\mathrm{d}s}\, \mathrm{E}\left[e^{-\int_t^T (Y_2(s)+\ldots+Y_d(s))\,\mathrm{d}s} \middle| \mathcal{F}(t)\right]. \end{aligned} \tag{4.5}$$

A similar calculation yields, using (4.4) and (4.3), yields

$$\begin{aligned} & e^{-\int_t^T \left(f_t^r(s)+f_t^2(s)+\ldots+f_t^d(s)\right) \mathrm{d}s} f_t^r(T) \\ & = e^{-\int_t^T g_t(s)\,\mathrm{d}s} g_t(T)\, \mathrm{E}\left[e^{-\int_t^T (Y_2(s)+\ldots+Y_d(s))\,\mathrm{d}s} \middle| \mathcal{F}(t)\right]. \end{aligned}$$

Dividing with the identity (4.5) above, we conclude that

$$f_t^r(T) = g_t(T),$$

which holds for all T where $t < T$, and we conclude that the *generalised forward rate* (on the left hand side) equals the forward interest rate (on the right hand side).

The calculations relied critically on the independence assumption, and in the general case the generalised forward rate for the interest is not equal to the forward interest rate. Intuitively, when the interest rate appears together with other dependent rates, the forward rates need to compensate for this dependence. Thus, the generalised forward rate includes a "covariance" term, which is not present in the usual forward interest rate.

4.2 Comparison With Other Dependent Setups

For the case of dependent affine rates, there have been other proposals for the definition of forward rates. In [11], the model contains an interest rate and a mortality rate which are dependent. This corresponds to the case $d = 2$, where $r(t) = Y_1(t)$ is the interest rate and $\mu(t) = Y_2(t)$ is the mortaltiy rate. Their approach is to keep the definition of the forward interest rate $g_t : [t, \infty) \to \mathbb{R}_+$ unchanged, and then find forward mortality rates that are compatible with this definition. In order to make this idea work, they define two different mortality rates, one for pure endowments, and one for term insurances. We briefly review this approach and compare to the definition of the generalised forward rates in the previous section.

The *forward mortality rate for pure endowments*, $h_t^{\mathrm{pe}} : [t, \infty) \to \mathbb{R}_+$, is defined as the function satisfying

$$\mathrm{E}\left[e^{-\int_t^T (r(s)+\mu(s))\,\mathrm{d}s} \middle| \mathcal{F}(t)\right] = e^{-\int_t^T \left(g_t(s)+h_t^{\mathrm{pe}}(s)\right) \mathrm{d}s}.$$

In terms of the generalised forward rates, f_t^r and f_t^μ, we can use the first part of (4.2) and write the forward mortality rate for pure endowment as,

$$h_t^{\text{pe}}(s) = f_t^r(s) + f_t^\mu(s) - g_t(s), \tag{4.6}$$

which in particular shows that it is well-defined. The forward mortality rate for pure endowment can be given an intuitive interpretation. Recall that the generalised forward rates are different from the usual definition of forward rates, because the mortality rate appears together with another dependent rate, thus the generalised forward rates contains a "covariance" part. The forward mortality rate for pure endowments corresponds to moving the "covariance" from the forward interest rate into the forward mortality rate, instead of having a part in each of the forward rates. In other words, $f_t^r + f_t^\mu$ contains the "covariance" terms, and subtracting g_t, which does not contain any "covariance" terms, the "covariance" terms are contained in h_t^{pe}. In this way, the original definition of the forward interest rate can be kept unaltered, but one can say that the forward mortality rate for pure endowment h_t^{pe} contains a "covariance" term belonging to the interest rate.

The *forward mortality rate for term insurances*, $h_t^{\text{ti}} : [t, \infty) \to \mathbb{R}_+$, is defined as the function satisfying,

$$\mathrm{E}\left[\int_t^T e^{-\int_t^u (r(s)+\mu(s))\,\mathrm{d}s} \mu(u)\,\mathrm{d}u \,\middle|\, \mathcal{F}(t)\right] = \int_t^T e^{-\int_t^u \left(g_t(s)+h_t^{\text{ti}}(s)\right)\mathrm{d}s} h_t^{\text{ti}}(u)\,\mathrm{d}u. \tag{4.7}$$

To establish that h_t^{ti} is well-defined is not as easy as with the forward mortality rate for the pure endowment. First, see that the definition depends on the choice of T. It is natural to make the assumption that the forward mortality rate for term insurances h_t^{ti} is independent of T. This assumption is implicity made in the notation used in [11], and the assumption is also made for the forward mortality rate for pure endowment. With this assumption of independence of T, we can differentiate with respect to T, and find the equivalent definition,

$$\mathrm{E}\left[e^{-\int_t^T (r(s)+\mu(s))\,\mathrm{d}s} \mu(T) \,\middle|\, \mathcal{F}(t)\right] = e^{-\int_t^T \left(g_t(s)+h_t^{\text{ti}}(s)\right)\mathrm{d}s} h_t^{\text{ti}}(T), \tag{4.8}$$

for $T \geq t$. For the rest of the paper, this definition is used.

Comparing to the generalised forward rates, we consider a policy consisting of a life annuity with a payment rate b, and a term insurance with payment 1 upon death. The policy terminates at time T. The expected present value at time t is

$$\mathrm{E}\left[\int_t^T e^{-\int_t^s (r(s)+\mu(s))\,\mathrm{d}s} \left(b + \mu(s)\right)\,\mathrm{d}s \,\middle|\, \mathcal{F}(t)\right]$$

$$= \int_t^T e^{-\int_t^s (f_t^{r:(r+\mu)}(s)+f_t^{\mu:(r+\mu)}(s))\,\mathrm{d}s} \left(b + f_t^{\mu:(r+\mu)}(s)\right) \mathrm{d}s$$
$$= \int_t^T e^{-\int_t^s g_t(s)\,\mathrm{d}s} \left(e^{-\int_t^s h_t^{\mathrm{pe}}(s)\,\mathrm{d}s} b + e^{-\int_t^s h_t^{\mathrm{ti}}(s)\,\mathrm{d}s} h_t^{\mathrm{ti}}(s)\right) \mathrm{d}s,$$

where we first wrote it in terms of the generalised forward rates, and afterwards in terms of the forward mortality rates for pure endowment and term insurances, respectively. This illustrates the difference between the different types of forward rates. The generalised forward rate for mortality can be used for both the life annuity and the term insurance, whereas with the other forward mortality rate definitions, one need a different one for a different product. If the interest rate is independent of the mortality rate, the different forward mortality rates simplify and they all equal the usual usual forward mortality rate.

4.2.1 Forward Mortality Rate for Term Insurances not-so-well Defined

We have not examined whether the forward mortality rate for term insurances is well defined. It turns out, that when there is dependence between the interest and mortality rates, there are cases where the forward mortality rate for term insurances defined by (4.8) does not exist. This will in particular be the case for models with positive correllation.

Consider a interest and mortality rate model $(r(t), \mu(t))$. This give us a set of generalised forward rates, and a forward mortality rate for pure endowment. Assume that the following assumptions hold.

Assumption 4.4. *Let a model for the interest and mortality rates $r(t)$ and $\mu(t)$ be given. The assumptions are,*

1. $h_t^{\mathrm{pe}}(s) > 0$ *for all* $s > t$.
2. h_t^{pe} *is bounded from below for some timepoint, i.e. there exist* $\varepsilon > 0$ *and* $t_0 > 0$ *such that* $h_t^{\mathrm{pe}}(s) > \varepsilon$ *for all* $s > t_0$.
3. *The forward interest rate is greater than the generalised forward rate for the interest,* $g_t(s) > f_t^r(s)$, *for all* $s > t$.

It is indeed possible to construct models where these assumptions hold, and indeed, they will hold for most models when there is a positive correlation between the interest rate and mortality rate. The first two assumptions state that the forward mortality rate for pure endowment is positive and bounded below from some time, which is satisfied in reasonable models. The third assumption assumption usually holds when there is a positive correlation between the interest and mortality rate.

The forward mortality rate for pure endowment, $h_t^{\mathrm{pe}}(s)$, present in the assumptions, is not the object of interest in this example. In view of (4.6), it can be thought of as a placeholder for $f_t^r + f_t^\mu - g_t$.

Proposition 4.5. *Under Assumption 4.4, there exists a $T > 0$ such that the forward mortality rate for term insurances $h_t^{\mathrm{ti}}(s)$ given by (4.8) does not exist for $s > T$.*

Proof. Combining (4.8) and (4.2), and then using (4.6) twice, we get that

$$\begin{aligned} e^{-\int_t^T h_t^{\mathrm{ti}}(s)\,\mathrm{d}s} h_t^{\mathrm{ti}}(T) &= e^{-\int_t^T \left(f_t^r(s)+f_t^\mu(s)-g_t(s)\right)\mathrm{d}s} f_t^\mu(T) \\ &= e^{-\int_t^T h_t^{\mathrm{pe}}(s)\,\mathrm{d}s} \left(h_t^{\mathrm{pe}}(T) + g_t(T) - f_t^r(T)\right), \end{aligned}$$

and by integration we find

$$e^{-\int_t^T h_t^{\mathrm{ti}}(s)\,\mathrm{d}s} = 1 - \int_t^T e^{-\int_t^\tau h_t^{\mathrm{pe}}(s)\,\mathrm{d}s} \left(h_t^{\mathrm{pe}}(\tau) + g_t(\tau) - f_t^r(\tau)\right) \mathrm{d}\tau. \tag{4.9}$$

Since the left hand side must be positive for any T, we conclude that the condition

$$\int_t^T e^{-\int_t^\tau h_t^{\mathrm{pe}}(s)\,\mathrm{d}s} \left(h_t^{\mathrm{pe}}(\tau) + g_t(\tau) - f_t^r(\tau)\right) \mathrm{d}\tau < 1 \tag{4.10}$$

is necessary for the forward mortality rate for term insurances to be well-defined.

Under the first assumption, the forward mortality rate for pure endowment, h_t^{pe}, defines a distribution in a two-state Markov chain, and we recognise the integral $\int_t^T e^{-\int_t^\tau h_t^{\mathrm{pe}}(s)\,\mathrm{d}s} h_t^{\mathrm{pe}}(\tau)\,\mathrm{d}\tau$ as a probability: Let Z be a stochastic variable that denotes the lifetime in a survival model where death occurs with rate $h_t^{\mathrm{pe}}(s)$ at time s. Then

$$\int_t^T e^{-\int_t^\tau h_t^{\mathrm{pe}}(s)\,\mathrm{d}s} h_t^{\mathrm{pe}}(\tau)\,\mathrm{d}\tau = P(Z \le T \mid Z > t).$$

Also, under the second assumption the probability converges to 1,

$$P(Z \le T \mid Z > t) \to 1 \text{ for } T \to \infty.$$

Consider now (4.10). Under the third assumption, $g_t(s) > f_t^r(s)$ for all $s > t$, there exists $\varepsilon > 0$ and $T^* \ge t$ such that

$$\int_t^T e^{-\int_t^\tau h_t^{\mathrm{pe}}(s)\,\mathrm{d}s} \left(g_t(\tau) - f_t^r(\tau)\right) \mathrm{d}\tau > \varepsilon,$$

for all $T > T^*$. This allows us to conclude, for a $T > T^*$ large enough, such that $P(Z \leq T \mid Z > t) > 1 - \varepsilon$, that

$$\int_t^T e^{-\int_t^\tau h_t^{\mathrm{pe}}(s)\,\mathrm{d}s} \left(h_t^{\mathrm{pe}}(\tau) + g_t(\tau) - f_t^r(\tau)\right) \mathrm{d}\tau > P(Z \leq T \mid Z > t) + \varepsilon > 1.$$

This contradicts (4.10), and the forward mortality rate for term insurances does not exist. □

For completeness, we specify a model satisfying Assumption 4.4. Let the 2-dimensional process $\mathbf{X}$ satisfy

$$\begin{aligned} \mathrm{d}X_1(t) &= (1 - X_1(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W_1(t), \\ \mathrm{d}X_2(t) &= (1 - X_2(t))\,\mathrm{d}t + \sigma\lambda\,\mathrm{d}W_1(t) + \sigma\sqrt{1-\lambda^2}\,\mathrm{d}W_2(t), \end{aligned}$$

with $X(0) = (1,1)^\top$. Let the interest rate and mortality rate be given by

$$\begin{aligned} r(t) &= r_0 X_1(t), \\ \mu(t) &= \mu^\circ(t) + X_2(t) - 1, \end{aligned}$$

with parameters $\lambda = 0.8$, $\sigma = 0.07$ and base mortality

$$\mu^\circ(t) = 5 \cdot 10^{-4} + 7.5858 \cdot 10^{-5} \cdot 1.09144^{50+t}.$$

That this model satisfies Assumption 4.4 is not shown here.

4.2.2 Somewhat Generalised Forward Rates

A discussion of the concept of forward rates, and generalisations, should not be undertaken without a reference to [14]. In the article, a sceptical view on the concept of forward mortality rates is adopted, and the fruitfulness of the concept is questioned. In view of this article, it is not at all clear that the concept of forward mortality rates (and generalised forward rates) is fruitful beyond being a convenient notation for the quantities needed for calculation of certain life insurance liabilites under a stochastic intensity assumption.

The article [14] also discusses requirements for more generalised forward rates. The generalised forward rates proposed in this article does not meet all the requirements set up in [14]. In particular, in life insurance models where one needs to use the relation (2.6), the generalised forward rates are not applicable. Thus, it would probably be more suiting to call them *somewhat generalised forward rates*.

5 Modelling Interest and Surrender

In order to illustrate the methods proposed, we put up a specific model for dependent interest and surrender rate. We model the interest rate as a stochastic diffusion process r, and the surrender rate by the diffusion process η. The interest and surrender rates are then modelled as dependent processes, within the affine setup presented in Section 2. Within the Solvency II regime, one is required to model surrender behaviour, and also take into consideration any dependence of the interest rate (i.e. the economic environment), see Section 3.5 in [5]. This model is thus an example of how this can be done.

5.1 Correlated Interest and Surrender Model

Let $\eta^0(t)$ be a deterministic surrender rate, corresponding to best estimate, i.e. the expectation of the future surrender rate. The interest rate $r(t)$ and surrender rate $\eta(t)$ are then modelled as an affine transformation of $\mathbf{X}$ of the form,

$$\begin{aligned} r(t) &= X_1(t), \\ \eta(t) &= \eta^0(t) X_2(t), \end{aligned}$$

where $\mathbf{X}$ is a 2-dimensional stochastic diffusion process. The process $\mathbf{X}$ satisfies the stochastic differential equation,

$$\begin{aligned} \mathrm{d}X_1(t) &= (b_1(t) - \beta_1 X_1(t))\,\mathrm{d}t + \sigma_1 \left(\sqrt{1-\rho^2}\,\mathrm{d}W_1(t) + \rho\sqrt{X_2(t)}\,\mathrm{d}W_2(t) \right), \\ \mathrm{d}X_2(t) &= (b_2 - \beta_2 X_2(t))\,\mathrm{d}t + \sigma_2\sqrt{X_2(t)}\,\mathrm{d}W_2(t), \end{aligned} \tag{5.1}$$

where $\mathbf{W}$ is a 2-dimensional standard Brownian motion. The parameters satisfy b_2, β_1, β_2, σ_1, $\sigma_2 \in \mathbb{R}_+$ and $\rho \in [-1, 1]$, and the function $b_1 : \mathbb{R}_+ \to \mathbb{R}$ is chosen such that an initial term structure is fitted.

The process $\mathbf{X}_2$ models relative deviations of the surrender rate from the best estimate, and it stays non-negative, hence the surrender rate $\eta(t)$ is non-negative. The interest rate process is a mix between a Hull-White Vašíček and a Heston model.

The model satisfies our criteria. It is affine, since $\mathbf{X}$ is affine and the surrender and interest rate is an affine transformation of $\mathbf{X}$. The surrender rate is non-negative. Also, choosing no, or little, mean reversion, stress scenarios produced by the model are close to parallel shifts of the forward rates, which resembles the stress scenarios of the standard model of Solvency II.

5.1.1 Correlation

The correlation between the interest rate and the surrender rate is not in general equal to the dependency parameter ρ. If we assume that $\mathrm{E}[X_2(t)] = 1$ for all t, we can calculate the correlation, using standard methods[2],

$$\mathrm{Corr}[r(t), \eta(t)] = \rho \frac{e^{(\beta_1+\beta_2)t} - 1}{\beta_1 + \beta_2} \sqrt{\frac{2\beta_1}{e^{2\beta_1 t} - 1}} \sqrt{\frac{2\beta_2}{e^{2\beta_2 t} - 1}}.$$

In the special case where $\beta_1 = \beta_2$, we get

$$\mathrm{Corr}[r(t), \eta(t)] = \rho.$$

When the parameters are chosen in Section 5.5.1 below, we see that indeed $\mathrm{E}[X_2(t)] = 1$ and $\beta_1 = \beta_2$ holds.

5.2 The (Life Insurance) Product

Consider a simple savings contract with a buy-back option. The savings contract consists of a guaranteed payment of 1 at retirement at time T. There is an account at the provider with a guaranteed interest rate $\hat{r}$ until time T. The value at time t of the account is then,

$$U(t) = e^{-\hat{r}(T-t)}. \tag{5.2}$$

The owner of the savings contract can then at any time before time T surrender the contract and receive the current account value $U(t)$.

The account value $U(t)$ is not necessarily identical to the reserve (market value) of the savings contract, thus the savings contract provider has a risk. In order to best estimate the value of the account, the surrender behaviour should be taken into account. There are different ways to valuate the surrender option, see [12] and references therein, and [10]. In this paper we adopt the intensity approach, and assume that the insured surrenders with rate $\eta(t)$ at time t, i.e. in a short time interval $[t, t + \mathrm{d}t]$, the insured surrenders with probability $\eta(t)\,\mathrm{d}t$, given that surrender has not occured before time t. We adopt the life insurance setup of Section 3, and consider the state of the insured in the state space $\mathcal{J}$ consisting of the two states *alive* and *surrendered*, corresponding to Figure 2.

This savings contract is a simplified version of the product considered in Example 3.1 and the Markov model shown in Figure 1. It is straightforward to include the mortality modelling done in Example 3.1, but to keep the notation simple, we omit it from this example.

[2] The quantities $\mathrm{E}[r(t)]$ and $\mathrm{E}[\eta(t)]$ can be found taking expectation on the Itô representation, and solving a differential equation. The expectation $\mathrm{E}[r(t)\eta(t)]$ can be found analogously, by first finding a stochastic differential equation for the process $t \mapsto r(t)\eta(t)$.

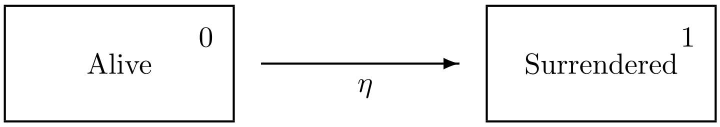

Figure 2: *Markov model for the surrender model.*

The payments of the contract consist of a single payment upon retirement at time T, and a payment upon surrender at time t of size $U(t)$. That is, the total payments $B(t)$ at time t is given by

$$\mathrm{d}B(t) = U(t)\,\mathrm{d}N_{01}(t) + 1_{(Z(t)=0)}\,\mathrm{d}\varepsilon_T(t),$$

where ε_T is the Dirac measure at T. Analogously to the calculations in Example 3.1 and Example 4.3, we find the present value at time t of the contract as

$$\begin{aligned} PV^{\mathbf{L}}(t) &= \int_t^T e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}B(s) \\ &= \int_t^T e^{-\int_t^s r(\tau)\,\mathrm{d}\tau} U(s)\,\mathrm{d}N_{01}(s) + e^{-\int_t^T r(\tau)\,\mathrm{d}\tau} 1_{(Z(t)=0)}, \end{aligned} \tag{5.3}$$

and the market value at time t is, given the savings contract has not been surrendered,

$$\begin{aligned} V(t) &= \mathrm{E}\left[PV^{\mathbf{L}}(t) \middle| \mathcal{F}(t), Z(t)=0 \right] \\ &= \mathrm{E}\left[\int_t^T e^{-\int_t^s (r(\tau)+\eta(\tau))\,\mathrm{d}\tau} \eta(s) U(s)\,\mathrm{d}s + e^{-\int_t^T (r(\tau)+\eta(\tau))\,\mathrm{d}\tau} \middle| \mathcal{F}^{\mathbf{X}}(t) \right] \\ &= \int_t^T e^{-\int_t^s (f_t^{r:(r+\eta)}(\tau) + f_t^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} f_t^{\eta:(r+\eta)}(s) U(s)\,\mathrm{d}s \\ &\quad + e^{-\int_t^T (f_t^{r:(r+\eta)}(\tau) + f_t^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau}. \end{aligned} \tag{5.4}$$

Here we used Remark 4.2. The notation used is introduced in Example 4.3 above.

5.3 Solvency II

For Solvency II purposes one wants to control the risk of default, such that it is less than 99.5% during the following year. In this section we specify how to interpret this in our setup, following the reasoning of Section 1.1 in [3].

We want to find the loss after one year, which is a stochastic variable, and find quantiles in the distribution of this stochastic variable. Let $PV(t)$ denote the present value at time t of future payments of the insurance company. At time 0, the Solvency II loss can be written as

$$\mathrm{E}\left[PV(0) \,| \mathcal{F}(1) \right],$$

where the expectation is taken using the market measure, or some reserving measure. For the rest of the paper, we refer to this measure as the market measure. For simplicity, we ignore the so-called unsystematic risk during the first year, that is, we take average of the Markov chain $\mathbf{Z}$, conditionally on the underlying intensities $\mathbf{X}$. Formally, we define the *Solvency II loss after 1 year* as

$$L = \mathrm{E}\left[PV(0)\,\middle|\mathcal{F}^{\mathbf{X}}(1)\right].$$

Both liabilities and assets must be taken into account, so the present value takes the form $PV(t) = PV^{\mathbf{L}}(t) - PV^{\mathbf{A}}(t)$, that is, the present value of the liabilities less the assets.

For the specific life insurance contract with present value (5.3), we obtain,

$$\begin{aligned}
&\mathrm{E}\left[PV^{\mathbf{L}}(0)\,\middle|\mathcal{F}^{\mathbf{X}}(1)\right]\\
&= \int_0^1 e^{-\int_0^s (r(\tau)+\eta(\tau))\,\mathrm{d}\tau}\eta(s)U(s)\,\mathrm{d}s\\
&\quad + e^{-\int_0^1 (r(s)+\eta(s))\,\mathrm{d}s}\int_1^T e^{-\int_1^s (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} f_1^{\eta:(r+\eta)}(s)U(s)\,\mathrm{d}s\\
&\quad + e^{-\int_0^1 (r(s)+\eta(s))\,\mathrm{d}s} e^{-\int_1^T (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau}.
\end{aligned} \tag{5.5}$$

The simplest possible asset allocation is to deposit all capital in a savings account, earning the risk free interest rate. In that case, the present value of the assets is deterministic and equals the amount invested today. If the value of the liabilities is invested, we have

$$PV^{\mathbf{A}}(0) = V(0).$$

For our case, $V(0)$ is given by (5.4). Combining, we get a Solvency II loss,

$$L = \mathrm{E}\left[PV^{\mathbf{L}}(0) - PV^{\mathbf{A}}(0)\middle|\,\mathcal{F}^{\mathbf{X}}(1)\right] = \mathrm{E}\left[PV^{\mathbf{L}}(0)\middle|\,\mathcal{F}^{\mathbf{X}}(1)\right] - V(0)$$

which is the difference between (5.5) and (5.4). Rearranging the terms slightly, we can write it on the form,

$$\begin{aligned}
L = &\int_0^1 e^{-\int_0^s (r(\tau)+\eta(\tau))\,\mathrm{d}\tau}\eta(s)U(s)\,\mathrm{d}s\\
&- \int_0^1 e^{-\int_0^s (f_0^{r:(r+\eta)}(\tau)+f_0^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} f_0^{\eta:(r+\eta)}(s)U(s)\,\mathrm{d}s\\
&+ e^{-\int_0^1 (r(s)+\eta(s))\,\mathrm{d}s}\int_1^T e^{-\int_1^s (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} f_1^{\eta:(r+\eta)}(s)U(s)\,\mathrm{d}s\\
&- \int_1^T e^{-\int_0^s (f_0^{r:(r+\eta)}(\tau)+f_0^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} f_0^{\eta:(r+\eta)}(s)U(s)\,\mathrm{d}s\\
&+ e^{-\int_0^1 (r(s)+\eta(s))\,\mathrm{d}s} e^{-\int_1^T (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau}\\
&- e^{-\int_0^T (f_0^{r:(r+\eta)}(\tau)+f_0^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau}.
\end{aligned}$$

The first two lines correspond to the losses arising during the first year because of incorrect expectations of interest and surrender behaviour. The last four lines correspond to changed expectations of the future, arising because of information received during the first year. That is, the third and fourth line corresponds to changed expectations of the future about the surrender payments, and the last two lines corresponds to changed expectations of the future about the payment occuring at retirement. Intuitively, the information received during the first year allows for an exact discounting during the first year, and a more precise valuation of the discounting and surrender behaviour occuring from year 1 an onwards.

The loss can be written in a simpler form. Using the notation that, for $s \leq t$, $f_t^{r:(r+\eta)}(s) = r(s)$ and $f_t^{\eta:(r+\eta)}(s) = \eta(s)$, we can write the Solvency II loss as

$$
\begin{aligned}
L = &\int_0^T e^{-\int_0^s (f_1^{r:(r+\eta)}(\tau) + f_1^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} f_1^{\eta:(r+\eta)}(s) U(s)\,\mathrm{d}s \\
&- \int_0^T e^{-\int_0^s (f_0^{r:(r+\eta)}(\tau) + f_0^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} f_0^{\eta:(r+\eta)}(s) U(s)\,\mathrm{d}s \\
&+ e^{-\int_0^T (f_1^{r:(r+\eta)}(\tau) + f_1^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} - e^{-\int_0^T (f_0^{r:(r+\eta)}(\tau) + f_0^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau}.
\end{aligned}
\tag{5.6}
$$

Recalling that $f_1^{r:(r+\eta)}$ and $f_1^{\eta:(r+\eta)}$ are $\mathcal{F}^{\mathbf{X}}(1)$ measurable, we can use that $\mathbf{X}$ is a Markov process and see that $f_1^{r:(r+\eta)}$ and $f_1^{\eta:(r+\eta)}$ are $r(1)$ and $\eta(1)$ measurable. Thus, the loss can be found by simulation of $r(s)$ and $\eta(s)$ for $0 \leq s \leq 1$. The simulation must be done under the real world probability measure. This is opposed to the market, or reserving, measure, that was used to find the loss. In this paper, we assume for simplicity that the two measures are identical, and do not adapt a change of measure approach, relieving us from discussions of preservation of the Markov property during measure changes.

5.4 Hedging Strategy with a Continuously Paid Coupon Bond

In practice, an insurer tries to hedge the interest rate risk, thereby reducing the loss significantly. We consider a simple static hedging strategy, in a bond with continuous coupon payments of the form,

$$
c(t) = e^{-\int_0^t f_0^{\eta:\eta}(\tau)\,\mathrm{d}\tau} f_0^{\eta:\eta}(t) U(t). \tag{5.7}
$$

For more details, see e.g. [12]. This corresponds to the expected payments of the life insurance contract, conditional on the interest rate. We can associate a payment process $\mathbf{A}$ with the bond, given by $\mathrm{d}A(t) = c(t)\,\mathrm{d}t$. The present value of future payments associated with the bond is then,

$$\begin{aligned} PV^{\mathbf{A}}(t) &= \int_t^T e^{-\int_t^s r(\tau)\,\mathrm{d}\tau}\,\mathrm{d}A(s) \\ &= \int_t^T e^{-\int_t^s r(\tau)\,\mathrm{d}\tau} e^{-\int_0^s f_0^{\eta:\eta}(\tau)\,\mathrm{d}\tau} f_0^{\eta:\eta}(s)U(s)\,\mathrm{d}s. \end{aligned}$$

This hedging strategy is the mean-variance optimal static hedging strategy when interest and surrender are independent. If there is a correlation between the interest and surrender rate, this strategy is not optimal. The mean-variance optimal static hedging strategy is in that case more complicated. These considerations are for simplicity omitted in this paper, and deferred for future studies.

We note that the sign of the payments $\mathbf{A}$ is opposite of $\mathbf{B}$, where the latter are payments to the insured, the former are payments to the insurer. Considering the life insurance contract and the hedging strategy together, we obtain a Solvency II loss,

$$\begin{aligned} L &= \mathrm{E}\left[\int_0^T e^{-\int_0^s r(\tau)\,\mathrm{d}\tau}(\,\mathrm{d}B(s) - \,\mathrm{d}A(s))\middle| \mathcal{F}^{\mathbf{X}}(1)\right] \\ &= \int_0^1 e^{-\int_0^s r(\tau)\,\mathrm{d}\tau}\left(e^{-\int_0^s \eta(\tau)\,\mathrm{d}\tau}\eta(s) - e^{-\int_0^s f_0^{\eta:\eta}(s)\,\mathrm{d}s} f_0^{\eta:\eta}(s)\right)U(s)\,\mathrm{d}s \\ &\quad + e^{-\int_0^1 (r(s)+\eta(s))\,\mathrm{d}s}\left(\int_1^T e^{-\int_1^s (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} f_1^{\eta:(r+\eta)}(s)U(s)\,\mathrm{d}s\right. \\ &\qquad \left. + e^{-\int_1^T (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau}\right) \\ &\quad - e^{-\int_0^1 (r(s)+f_0^{\eta:\eta}(s))\,\mathrm{d}s}\left(\int_1^T e^{-\int_1^s (f_1^{r:r}(\tau)+f_0^{\eta:\eta}(\tau))\,\mathrm{d}\tau} f_0^{\eta:\eta}(s)U(s)\,\mathrm{d}s\right. \\ &\qquad \left. + e^{-\int_1^T (f_1^{r:r}(\tau)+f_0^{\eta:\eta}(\tau))\,\mathrm{d}\tau}\right) \\ &= \int_0^T e^{-\int_0^s (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} f_1^{\eta:(r+\eta)}(s)U(s)\,\mathrm{d}s \\ &\quad - \int_0^T e^{-\int_0^s (f_1^{r:r}(\tau)+f_0^{\eta:\eta}(\tau))\,\mathrm{d}\tau} f_0^{\eta:\eta}(s)U(s)\,\mathrm{d}s \\ &\quad + e^{-\int_0^T (f_1^{r:(r+\eta)}(\tau)+f_1^{\eta:(r+\eta)}(\tau))\,\mathrm{d}\tau} - e^{-\int_0^T (f_1^{r:r}(\tau)+f_0^{\eta:\eta}(\tau))\,\mathrm{d}\tau}, \end{aligned} \tag{5.8}$$

Similar to (5.6), for $s \leq t$, the notation that $f_t^{r:(r+\eta)}(s) = f_t^{r:r}(s) = r(s)$ and $f_t^{\eta:(r+\eta)}(s) = \eta(s)$ is used for the last equality.

5.5 Numerical Results

In this section we numerically show some consequenses of modelling interest and surrender as positively correlated processes. First, the model is specified by choosing a set of parameters, partly inspired by the stress levels in the Solvency II Standard Formula. With this model, we examine the consequenses for the balance sheet value of the liabilities, and the level of the Solvency II capital requirement, that is, the liabilities in 1 years time.

For the Solvency II capital requirement, in practice in the industry, when there is no hedging, most of the risk is interest rate risk. Luckily, both in theory and practice, a lot of this can be hedged by e.g. buying bonds. For the numerical illustrations of the Solvency II capital requirement, we consider two different strategies for the assets, corresponding to the two strategies considered in Section 5.3 and Section 5.4, respectively. First, we consider the case where the interest rate risk is not hedged, and all assets are accumulated by the risk free interest rate. Second, we consider the case where the insurer tries to hedge the interest rate risk, and performs a static hedge.

5.5.1 Parameters

The numerical examples with the model (5.1) are carried out for different level of correlation, namely $\rho \in \{0, 0.3, 0.7\}$. Also, we consider two different guaranteed interest rates, namely $\hat{r} \in \{1\%, 4\%\}$. This corresponds to a low interest rate, which could be for a newly issued policy, and a high interest rate, which could be for a policy issued years ago, when the interest rate level was higher. We note that the base deterministic surrender rate η^0 corresponds to a person aged 40, thus with $T = 25$, the contract ends at age 65.

The parameters chosen for the interest and surrender rates are listed in Table 1, and in Figure 3 some realisations of the interest and surrender rates are shown. The initial value $X_1(0)$ and function $b_1(t)$ are chosen such that the term structure provided by the Danish FSA at August 17, 2012 is matched. Let $f^{\mathrm{FSA}}(t)$ denote the forward rate provided by the Danish FSA. Then the parameters X_1 and b_1 are fitted such that

$$\mathrm{E}\left[e^{-\int_0^t r(s)\,\mathrm{d}s}\right] = e^{-\int_0^t f^{\mathrm{FSA}}(s)\,\mathrm{d}s},$$

for all $t \geq 0$. The parameters of the model correspond to the measure used for valuating the market value of the life insurance liabilities. Thus, with respect to the interest rate it is the risk neutral measure. For simplicity, we assume that this measure equals the real world probability measure.

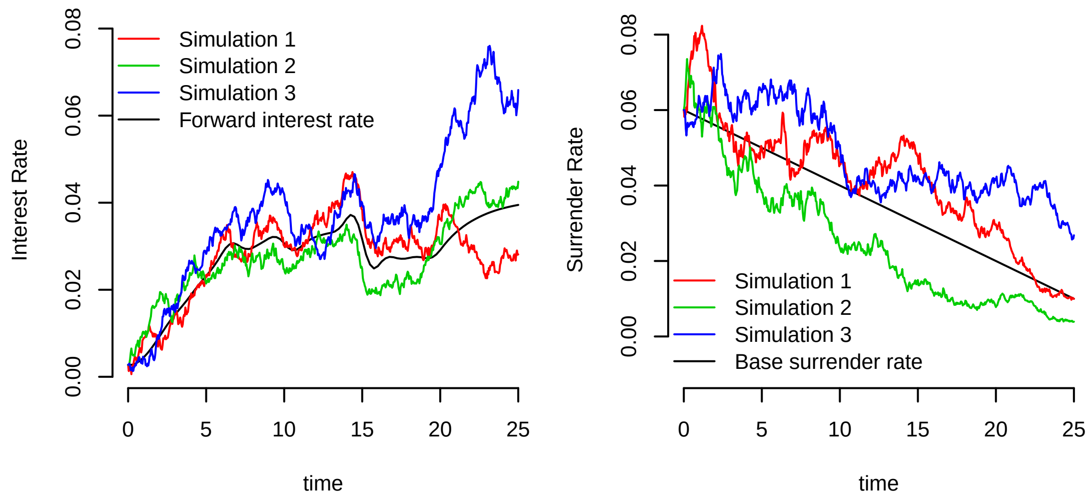

Figure 3: *Illustrative realisations of the interest rate (left) and the surrender rate (right), with* $\rho = 0.7$.

β_1	0.02	b_2	0.02	$\eta^0(t)$	$0.06 - 0.002 \cdot t$
σ_1	0.005	β_2	0.02	$X_2(0)$	1
		σ_2	0.15		

Table 1: *Parameters for correlated interest and surrender modelling. The initial value* $X_1(0)$ *and the function* $b_1(t)$ *are chosen such that the interest rate model matches the term structure provided by the Danish FSA for valuating life insurance liabilites, at August 17, 2012.*

5.5.2 Generalised Forward Rates

In Figure 4, the generalised forward rates are shown. They are calculated by solving the differential equations (2.3) and (2.5) numerically. For the interest rate, the forward interest rate supplied by the Danish FSA, f^{FSA}, is shown as well. We see that for the case $\rho = 0$ the generalised forward interest rate f_0^r is identical to the forward rate provided by the Danish FSA. This is as expected, since in the case $\rho = 0$ the interest rate and surrender rate are independent, and in this case the generalised forward rates are equal to the usual forward rates. For a positive correlation, the generalised forward rates are smaller. This is because the stochastic variable, $e^{-\int_0^t (r(s)+\eta(s))\,\mathrm{d}s}$, which is used to construct the generalised forward rates, has a heavier tail when the correlation is strictly positive, due to the exponential function. Intuitively, there is less diversification between the interest and surrender rate.

For the surrender rate, the basic deterministic surrender rate η^0 is shown as well as the generalised forward rates. Even though $\mathrm{E}\left[\eta(t)\right] = \eta^0(t)$, we see that the generalised

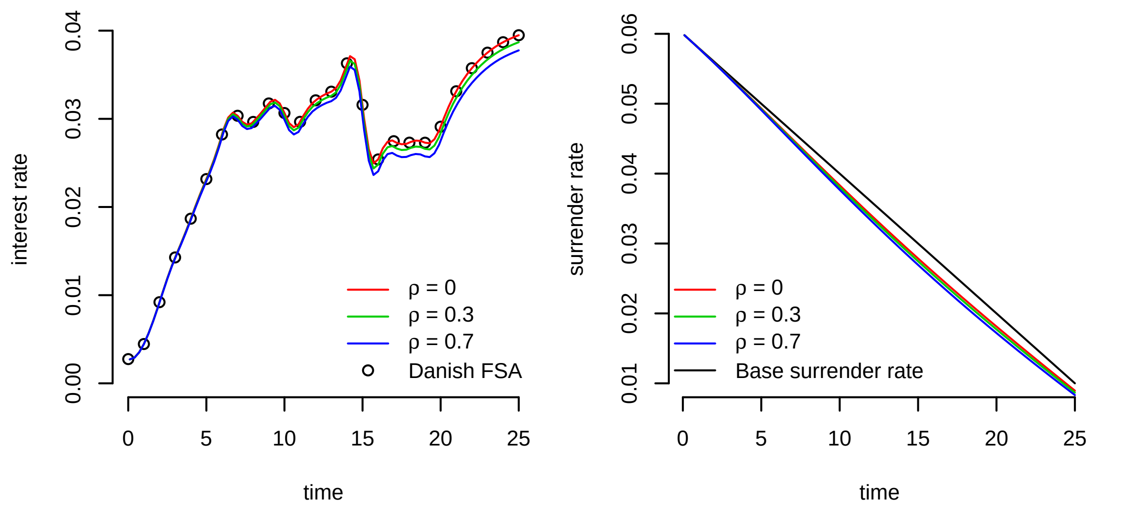

Figure 4: *Generalised forward rates. Left: for the interest rate, $f_0^{r:(r+\eta)}(t)$. Right: for the surrender rate, $f_0^{\eta:(r+\eta)}(t)$. The generalised forward rates are shown for different values of ρ. The forward interest rate extracted from the Danish FSA at August 17, 2012 is also shown, as well as the base deterministic surrender rate η^0. Higher values of ρ lead to lower values of the forward rates, corresponding to less discounting.*

forward rates are systematically lower than η^0. This is due to Jensens inequality, and to see this, consider the case $\rho = 0$, where we get,

$$\begin{aligned} e^{-\int_0^t (f_0^r(s)+f_0^\eta(s))\,\mathrm{d}s} &= \mathrm{E}\left[e^{-\int_0^t (r(s)+\eta(s))\,\mathrm{d}s}\right] \\ &= \mathrm{E}\left[e^{-\int_0^t r(s)\,\mathrm{d}s}\right] \mathrm{E}\left[e^{-\int_0^t \eta(s)\,\mathrm{d}s}\right] \\ &> \mathrm{E}\left[e^{-\int_0^t r(s)\,\mathrm{d}s}\right] e^{-\int_0^t \mathrm{E}[\eta(s)]\,\mathrm{d}s} \\ &= e^{-\int_0^t f_0^r(s)\,\mathrm{d}s} e^{-\int_0^t \eta^0(s)\,\mathrm{d}s}, \end{aligned}$$

for $t > 0$, using that the usual forward rate is identical to the generalised forward rate for $\rho = 0$. From this inequality, we obtain,

$$f_0^\eta(t) < \eta^0(t),$$

which is what was observed as the red and black lines in Figure 4. If there is a positive correlation, the generalised forward surrender rate, $f^{\eta:(r+\eta)}$, is even smaller, similar to the observation for the interest rates.

5.5.3 Market Value

The market value at time 0, $V(0)$ from (5.4), can be calculated, solving the integral numerically. For this, first use (4.1) to get

$$V(t) = \int_t^T e^{\phi(t,s)+\psi(t,s)^\top X(t)} f_t^{\eta:(r+\eta)}(s) U(s)\,\mathrm{d}s + e^{\phi(t,T)+\psi(t,T)^\top X(t)},$$

which is easier to handle from a computational point of view, because the functions ϕ and ψ are obtained in the process of calculating the generalised forward rates $f^{r:(r+\eta)}$ and $f^{\eta:(r+\eta)}$ when solving (2.3) and (2.5). The market value $V(0)$, dependent upon the guaranteed interest rate $\hat{r}$ and the correlation ρ, is shown in Table 2. The market values can be compared to the value of the policyholders account which is paid out on surrender. This is given by (5.2), calculated using the guaranteed interest rate. The value at time 0 is presented in Table 3.

		$\hat{r}$	
		4%	1%
	0	0.4567	0.6167
ρ	0.3	0.4595	0.6191
	0.7	0.4631	0.6222

Table 2: *Market value at time 0, $V(0)$, of the life insurance contract. The value is shown using three different correlations, corresponding to three different sets of generalised forward rates, red, green and blue from Figure 4. Two different levels of guaranteed interest rate, $\hat{r}$, is used, which leads to different surrender payouts $U(t)$.*

$\hat{r}$	
4%	1%
0.3679	0.7788

Table 3: *Initial value of the policyholders account, $U(0)$. For the high guaranteed interest rate (4%), the value is lower than the market value from Table 2. For the low guaranteed interest rate (1%), the value is higher than the market value.*

The market value without surrender modelling, calculated setting the surrender rate equal to zero, is 0.5037. It is independent of the guaranteed interest rate. From Table 2 it is seen that when we include surrender modelling the market value is somewhere between the value of the policyholders account and the market value calculated without surrender modelling.

For both cases of guaranteed interest rates, the market value increases with correlation. When we discussed the generalised forward rates in Section 5.5.2, we saw that the generalised forward rates decrease with increasing correlation, which is basically due to the convexity of the exponential function and Jensen's inequality. A smaller generalised forward interest rate leads to an increasing market value. For the surrender rate, it is more complicated. For the case of a guaranteed interest rate of 4%, an increase in the generalised forward surrender rate rate leads to a decrease in the market value, because the market value come closer to the value paid out on surrender. For the case of a guaranteed interest rate of 1%, the same argument tells us that an increase in the generalised forward surrender rate instead leads to an increasing market value. We see that the effect of the decreasing generalised forward interest rate is largest, and in total, for both levels of guaranteed interest rate, the market value increases when the correlation increases.

5.5.4 Solvency II

We examine the effect on the Solvency II capital requirement with two different strategies for the assets. For both strategies, the initial market value of the assets equals the market value of the liabilities. The first strategy is no hedging, and the second strategy is a simple static hedging strategy. This corresponds to the two strategies discussed in Section 5.3 and Section 5.4, respectively. For the first strategy, where all assets are invested in the bank account, the Solvency II loss is given by (5.6). For the second strategy, where the interest rate risk is hedged statically in a bond with continuous payments, the Solvency II loss is given by (5.8).

		No Hedge		Hedge	
		$\hat{r}$		$\hat{r}$	
		4%	1%	4%	1%
	0	0.069	0.077	0.014	0.025
ρ	0.3	0.072	0.072	0.018	0.030
	0.7	0.078	0.060	0.024	0.034

Table 4: *Simulated Solvency II loss. Without hedging it is given by* (5.6) *and with the hedging strategy it is given by* (5.8). *Applying an interest hedging strategy significantly lowers the Solvency II loss. Also, modelling correlation between interest and surrender has a significant impact on the Solvency II loss.*

In Table 4 the Solvency II loss for the different cases of hedging strategy, guaranteed interest rate risk and correlation is presented. It is immediately seen, that trying to hedge the interest rate risk by applying the simple hedging strategy significantly reduces the Solvency II loss.

For the case of no hedging strategy, we see two different correlation effects. When the guaranteed interest rate is 4%, a higher correlation means a higher Solvency II loss,

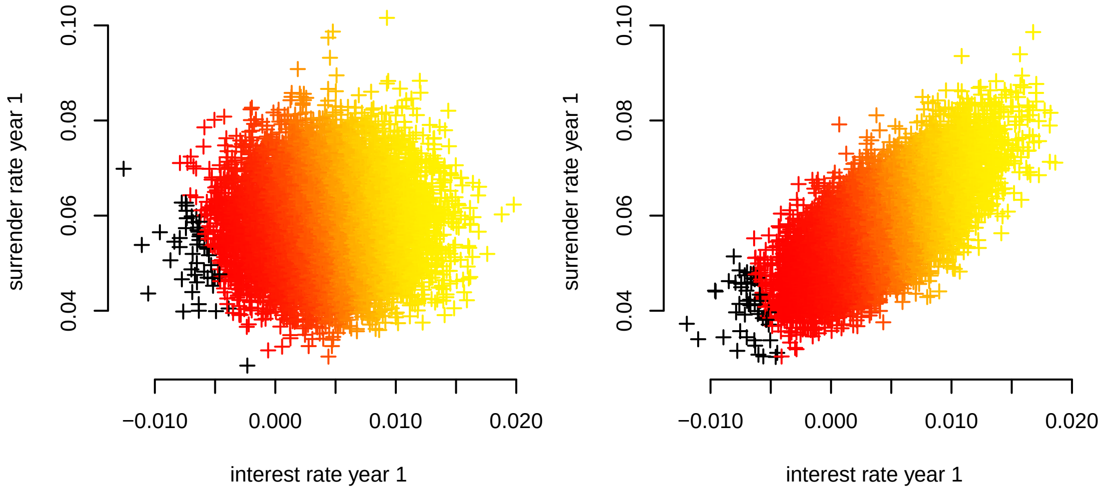

Figure 5: *Guaranteed interest rate 4%. Plot of the interest and surrender rate simulations after 1 year in the case without any hedging strategy and correlation $\rho = 0$ (left) and $\rho = 0.7$ (right). The color of a mark indicates the Solvency II loss* (5.6)*, where a darker color is a higher loss, and black colors are losses beyond the 99.5% quantile.*

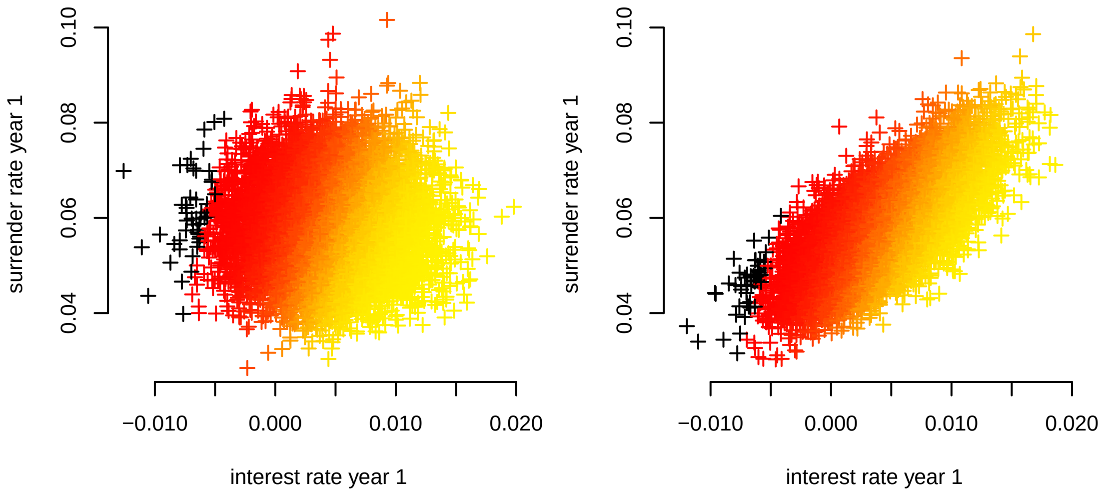

Figure 6: *Guaranteed interest rate 1%. Plot of the interest and surrender rate simulations after 1 year in the case without any hedging strategy and correlation $\rho = 0$ (left) and $\rho = 0.7$ (right). The color of a mark indicates the Solvency II loss* (5.6)*, where a darker color is a higher loss, and black colors are losses beyond the 99.5% quantile.*

because a decrease in interest and surrender rate both increase the present value of the contract payments. This is depicted in Figure 5, where we see that the loss increases with both decreasing interest and decreasing surrender. A higher correlation means that the probability of simultaneous drops in the interest and surrender rate occurs simultaneously, which can be seen at the right graph in Figure 5. When the guaranteed interest rate is instead 1%, a decrease in the surrender rate now means that Solvency II loss decrease, which can be seen in Figure 6. Introducing a correlation leads to less observations with decreasing interest and increasing surrender, thus leading to more diversification and reducing the Solvency II loss. This can be seen in the right graph of Figure 6.

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References

[1] Enrico Biffis. Affine processes for dynamic mortality and actuarial valuations. *Insurance: Mathematics & Economics*, 37:443–468, 2005.

[2] Matthias Börger. Deterministic Shock vs. Stochastic Value-at-Risk – An Analysis of the Solvency II Standard Model Approach to Longevity Risk. *Blätter der DGVFM*, 31 (2):225–259, 2010.

[3] Kristian Buchardt. Affine Processes in Life Insurance Mathematics - With a View Towards Solvency II. `http://math.ku.dk/~buchardt`, 2011. Thesis for the Master degree in Insurance Mathematics, Department of Mathematical Sciences, University of Copenhagen.

[4] Kristian Buchardt. Continuous Affine Processes: Transformations, Markov Chains and Life Insurance. (Working Paper), Department of Mathematical Sciences, University of Copenhagen, `http://math.ku.dk/~buchardt`, 2012.

[5] CEIOPS. CEIOPS' Advice for Level 2 Implementing Measures on Solvency II: Technical provisions, Article 86 a, Actuarial and statistical methodologies to calculate the best estimate. `https://eiopa.europa.eu/publications/sii-final-l2-advice/index.html`, 2009.

[6] Mikkel Dahl, Martin Melchior, and Thomas Møller. On systematic mortality risk and risk-minimization with survivor swaps. *Scandinavian Actuarial Journal*, 2-3:114–146, 2008.

[7] Mikkel Dahl and Thomas Møller. Valuation and Hedging of Life Insurance Liabilities with Systematic Mortality Risk. *Insurance: Mathematics & Economics*, 39 (2):193–217, 2006.

[8] Darrell Duffie, Jun Pan, and Kenneth Singleton. Transform Analysis and Asset Pricing for Affine Jump-Diffusions. *Econometrica*, 68, No. 6:1343–1376, 2000.

[9] Damir Filipovic. *Term Structure Models*. Spring Finance. Springer, 2009.

[10] Domenico De Giovanni. Lapse Rate Modeling: A Rational Expectation Approach. Finance Research Group Working Papers, University of Aarhus, Aarhus School of Business, Department of Business Studies, 2007.

[11] Kristian R. Miltersen and Svein-Arne Persson. Is Mortality Dead? Stochastic Forward Force of Mortality Rate Determined by No Arbitrage. (Working Paper), Norwegian School of Economics and Business Administration, 2005.

[12] Thomas Møller and Mogens Steffensen. *Market-Valuation Methods in Life and Pension Insurance*. International Series on Actuarial Science. Cambridge University Press, 2007.

[13] Ragnar Norberg. A theory of bonus in life insurance. *Finance and Stochastics*, 3:373–390, 1999.

[14] Ragnar Norberg. Forward mortality and other vital rates - Are they the way forward? *Insurance: Mathematics & Economics*, 47 (2):105–112, 2010.

[15] Ragnar Norberg. Optimal hedging of demographic risk in life insurance. *Finance and Stochastics*, pages 1–26, 2012.