

A Subordinated Markov Model for Stochastic Mortality

Xiaoming Liu

Department of Statistical & Actuarial Sciences
University of Western Ontario
(recently renamed as **Western University**)

May 22, 2013 at ASTIN, The Hague

Outline

Background on PH models in survival analysis

- PH models: Definition and properties

- Applications as an exploratory statistical model

Aging process and PH mortality model

- Proposed aging process and PH mortality model

- Model fitting results based on PH mortality model

Stochastic mortality modelling under PH model framework

- Introduce stochasticity by a subordinating Gamma process

- Measure change under the PH model framework

Future works and potential areas of application

Phase-type distribution— Definition

Consider a continuous-time **Markov process** with:

- ▶ finite state space $S = \{1, \dots, n\} \cup \Delta$, where Δ is **absorbing** and the other states are transient.
- ▶ initial distribution $(\alpha, 0)$
- ▶ infinitesimal generator (transition rate matrix)

$$\begin{pmatrix} \Lambda & \mathbf{q} \\ \mathbf{0} & 0 \end{pmatrix}, \quad \text{where } \mathbf{q} = -\Lambda \mathbf{e}$$

Phase-type distribution— Definition

Consider a continuous-time **Markov process** with:

- ▶ finite state space $S = \{1, \dots, n\} \cup \Delta$, where Δ is **absorbing** and the other states are transient.
- ▶ initial distribution $(\alpha, 0)$
- ▶ infinitesimal generator (transition rate matrix)

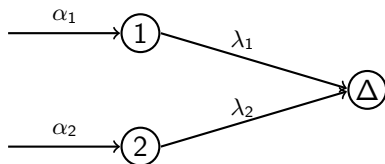
$$\begin{pmatrix} \Lambda & \mathbf{q} \\ \mathbf{0} & 0 \end{pmatrix}, \quad \text{where } \mathbf{q} = -\Lambda \mathbf{e}$$

Let X denote the time till absorption, then X has a **phase-type (PH)** distribution with a representation (α, Λ) of order n .

PH distribution: an extension of exponential distr.

The family of PH distribution contains:

- ▶ Mixture of exponentials

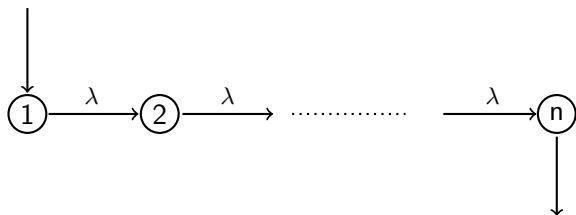


$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2), \quad \mathbf{\Lambda} = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}$$

- ▶ It is applied in queueing theory to model the service time distribution.

PH distribution: an extension of exponential distr.

- ▶ It also contains **Gamma distribution**

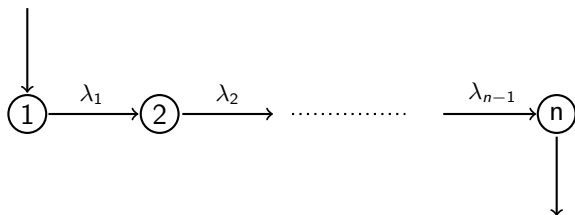


$$\alpha = (1, 0, \dots, 0), \quad \mathbf{\Lambda} = \begin{pmatrix} -\lambda & \lambda & 0 & \dots & 0 \\ 0 & -\lambda & \lambda & \dots & 0 \\ 0 & 0 & -\lambda & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\lambda \end{pmatrix}$$

- ▶ Gamma is well-known as a more general waiting time distribution.

Examples of PH models in survival analysis

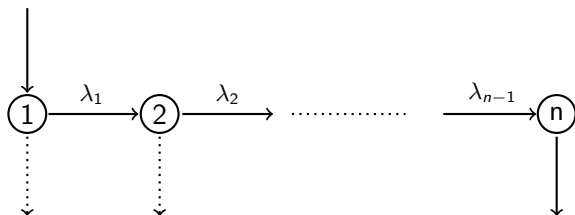
... as a flexible **latent-class** process with interpretable components:



- ▶ Engineering reliability models of **failure**: cumulative damage such as crack-spreading or corrosion.
- ▶ Biomedical models of **cancer incidence and development**: threshold crossings (CD4+), cell mutations, etc.
- ▶ Pharmacokinetic compartmental models of **drug duration**: sojourns through a succession of organs.
- ▶ Actuarial models of **illness (disabled)-death**: degree of illness or disability.

Examples of PH models in survival analysis

... as a flexible **latent-class** process with interpretable components:



- ▶ Engineering reliability models of **failure**: cumulative damage such as crack-spreading or corrosion.
- ▶ Biomedical models of **cancer incidence and development**: threshold crossings (CD4+), cell mutations, etc.
- ▶ Pharmacokinetic compartmental models of **drug duration**: sojourns through a succession of organs.
- ▶ Actuarial models of **illness (disabled)-death**: degree of illness or disability.

The key features underneath those modelling approach

- ▶ The happening of an event is explained as a process of going through a sequence of time-homogeneous Markovian transitions between discrete states.
- ▶ Even when the intermediate states are **latent or nonderterministic**, this type of the models is still very useful in providing intuitions about mechanisms to reflect distinct objectives.
- ▶ The **added levels of detail** can be supported by the corresponding scientific theory which in turn can provide new perspective to collect data or conduct experimental test.
- ▶ Generally, the parameters of the duration distribution are of primary interest. Nice mathematical tractability of PH models makes it feasible to obtain those parameters.
- ▶ More examples can be found in Moolgavkar (2004).

Matrix-Analytical Calculation

Let X be phase-type with representation $(\boldsymbol{\alpha}, \boldsymbol{\Lambda})$ of order n . Then:

- ▶ Distribution function

$$F(x) = 1 - \boldsymbol{\alpha} e^{\boldsymbol{\Lambda}x} \mathbf{e}, x > 0$$

- ▶ Density function

$$f(x) = F'(x) = \boldsymbol{\alpha} e^{\boldsymbol{\Lambda}x} \mathbf{q}, x > 0$$

- ▶ Laplace-Stieltjes transform

$$\psi(s) = \boldsymbol{\alpha}(s\mathbf{I} - \boldsymbol{\Lambda})^{-1} \mathbf{q}$$

- ▶ Non-central moments

$$m_k = (-1)^k k! \boldsymbol{\alpha} \boldsymbol{\Lambda}^{-k} \mathbf{e}, k = 1, 2, \dots$$

The matrix-exponential $e^{\mathbf{K}}$ is defined by the standard series expansion $\sum_{m=0}^{\infty} \mathbf{K}^m / m!$.

References: Neuts(1981) and Asmussen (1987).

Other Properties of Phase-type Distributions

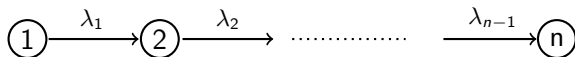
- ▶ Dense in the class of distributions with nonnegative support
- ▶ Preserve the underlying Markovian structure of stochastic models.
- ▶ Structurally informative with intuitive diagram representation
- ▶ Mathematically tractable

Problems with Phase-type distributions

- ▶ Non-uniqueness of representation
- ▶ Fitting problem is highly non-linear
- ▶ Large number of parameters to be estimated

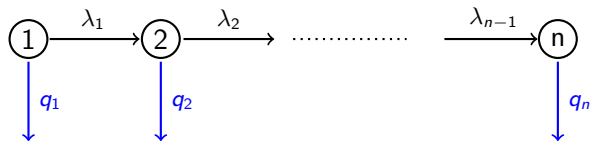
Aging process and a new mortality modelling approach

The proposed aging process and mortality modelling



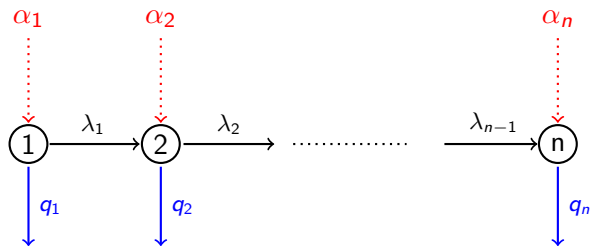
- ▶ This aging process is an internal process of deterioration in terms of human body's physiological capacity.
- ▶ The status of an individual's physiological capacity is called **physiological age or health index** in our papers. **The increased physiological age implies the increased susceptibility to various hazards.**

The proposed aging process and mortality modelling



- ▶ This aging process is an internal process of deterioration in terms of human body's physiological capacity.
- ▶ The status of an individual's physiological capacity is called **physiological age or health index** in our papers. **The increased physiological age implies the increased susceptibility to various hazards.**
- ▶ The increasing death rates with age is a reflection of the aging process; but death can be caused by other events too, e.g. an accident.

The proposed aging process and mortality modelling



- ▶ This aging process is an internal process of deterioration in terms of human body's physiological capacity.
- ▶ The status of an individual's physiological capacity is called **physiological age or health index** in our papers. **The increased physiological age implies the increased susceptibility to various hazards.**
- ▶ The increasing death rates with age is a reflection of the aging process; but death can be caused by other events too, e.g. an accident.

Main properties of the proposed model

- ▶ We focus on the relationship between mortality and physiological capacity (i.e. **physiological age or health index** in our papers), rather than a direct relationship between mortality and age.
- ▶ The model incorporates random nature in individual's biological aging process.
- ▶ The model allows other type of causes of death (other than aging) to be considered separately and additively in the stochastic aging mechanism.
- ▶ The time-till-death r.v. T is the time-till-absorption of the process, which is easy to calculate, **utilizing the matrix-analytic tools**.

Findings from the research on aging

- ▶ Various functions characterizing **physiological capacities** in living organisms decline in a way more or less **linearly** with age; note they are not declining in **exponential** form as the Gompertz law might suggest.
- ▶ The change of those physiological functions presents **variation** among individuals of a cohort, and this variation increases somewhat with age;
- ▶ **Generalized Definition (Collatz, 1986)**
Aging process is the **unidirectional and irreversible** course of intrinsic events which leads the organism from the beginning to the end of its life.

Physiological Capacities Tested by Bafitis and Sargent (1977)

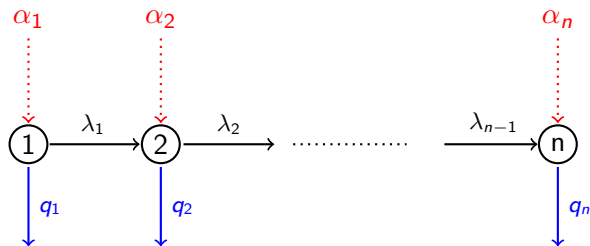
- ▶ Maximum Breathing capacity (MBC)
- ▶ Vital capacity (VC)
- ▶ Glomerular filtration rate (GFR)
- ▶ Renal plasma flow (RPF)
- ▶ Basal Metabolic rate (BMR)
- ▶ Conduction velocity (Ulnar nerve) (CV)
- ▶ Cardiac index (CI)
- ▶ Intracellular Water (ICW)

Yashin, A. I., Arbeev, K. G., Ukraintseva, S. V., Akushevich, I., and Kulminski, A. (2011), Patterns of Aging-Related Changes on the Way to 100: An Approach to Studying Aging, Mortality and Longevity From Longitudinal Data, *Monograph of the SOA Living to 100 Symposium*.

References on (human) aging

- ▶ Austad, S. N. 1997. *Why We Age: What Science Is Discovering about the Body's Journey through Life*. John Wiley & Sons, New York.
- ▶ Collatz, K.-G. 1986 Towards a Comparative Biology of Aging. In *Insect Aging*, K.-G. Collatz and R.S. Sohal, Eds. Berlin: Springer-Verlag.
- ▶ Featherman, D. L. 1986. Marker of Aging. *Research on Aging* 8(3): 339-365.
- ▶ Bafitis, H. and Sargent, F. 1977. Human Physiological Adaptability Through the Life Sequence. *Journal of gerontology* 32(4): 402-410.
- ▶ Yashin, A. I., Arbeev, K. G., Ukraintseva, S. V., Akushevich, I., and Kulminski, A. (2011), Patterns of Aging-Related Changes on the Way to 100: An Approach to Studying Aging, Mortality and Longevity From Longitudinal Data, *Monograph of the SOA Living to 100 Symposium*.

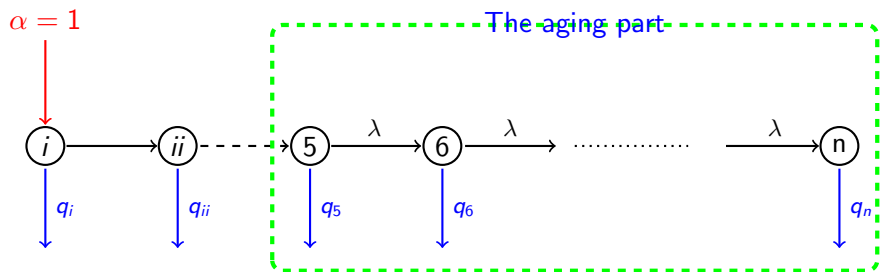
(α, Λ) in general aging model



$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n).$$

$$\Lambda = \begin{pmatrix} -(\lambda_1 + q_1) & \lambda_1 & 0 & \dots & 0 \\ 0 & -(\lambda_2 + q_2) & \lambda_2 & \dots & 0 \\ 0 & 0 & -(\lambda_3 + q_3) & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -q_n \end{pmatrix}$$

The proposed model for Swedish male (cohort) data



$$\lambda_i = \lambda, \text{ for } i = 5, \dots, 200,$$

$$q_i = \begin{cases} b + a + i^p \cdot q & \text{for } i_1 < i \leq i_2 \\ b + i^p \cdot q & \text{otherwise} \end{cases}$$

- ▶ Parameter b represents a general **b**ackground death rate.
- ▶ Parameter a represents the extra **a**ccidental death rate.
- ▶ Part $i^p * q$ represents an increasing deterioration pattern of physiological capacities due to aging.

Estimation method

- ▶ Weighted least square is used. The function minimized is

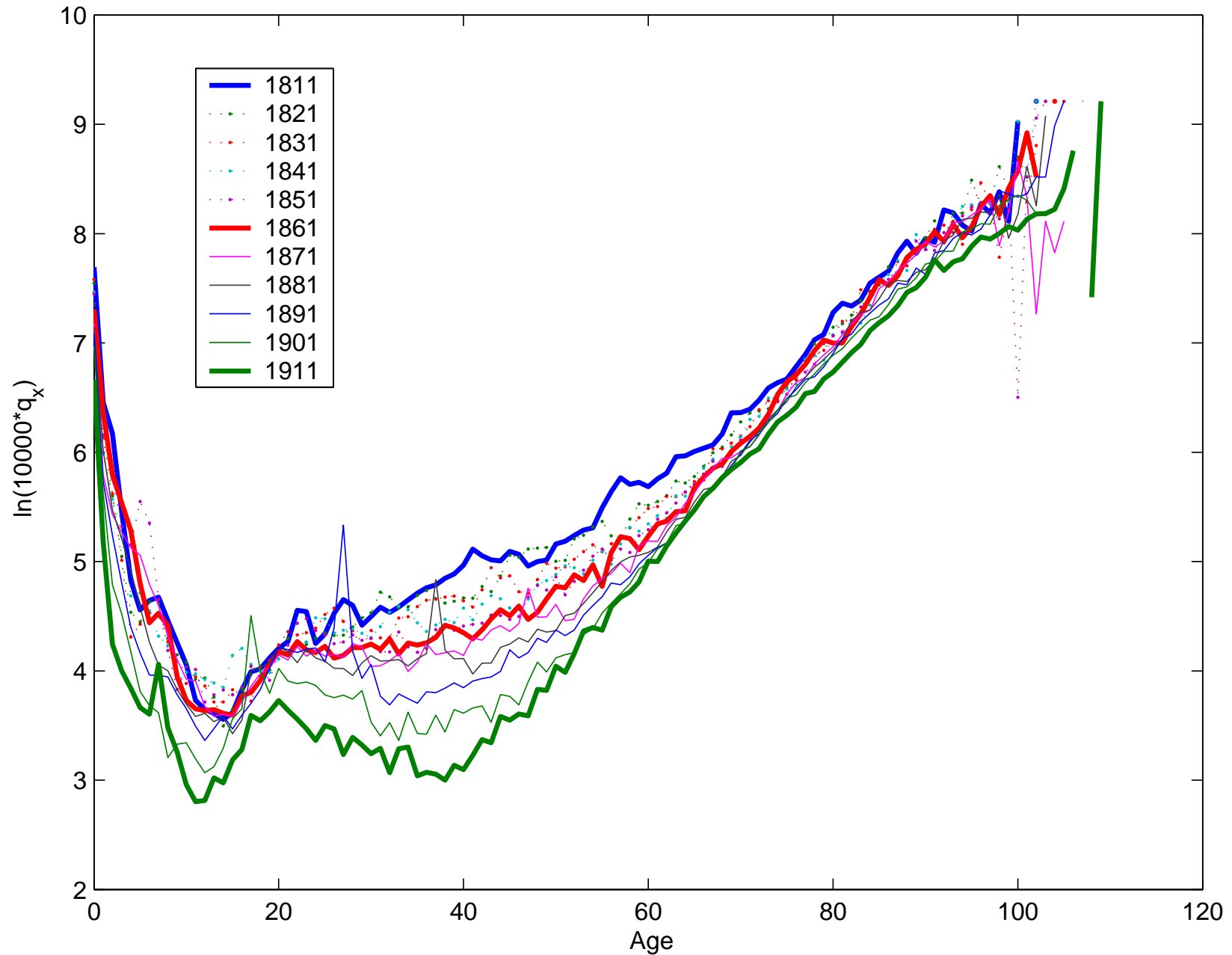
$$F = \sum_{x=0}^{\omega} (q_x - \hat{q}_x)^2 S(x)$$

Where q_x and $S(x)$ is the observed conditional death rate and survival value at age x , and \hat{q}_x is conditional death rate calculated by:

$$\begin{aligned}\hat{S}(x) &= \alpha e^{\Lambda x} e \\ \hat{q}(x) &= 1 - \frac{\hat{S}(x+1)}{\hat{S}(x)}\end{aligned}$$

- ▶ Parameters are obtained numerically by the Nelder-Mead simplex algorithm.

Mortality data for Sweden cohorts 1811 to 1911

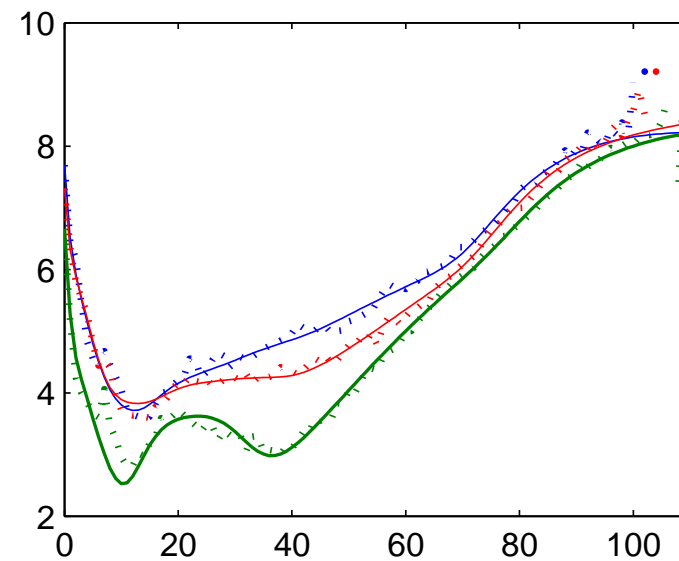
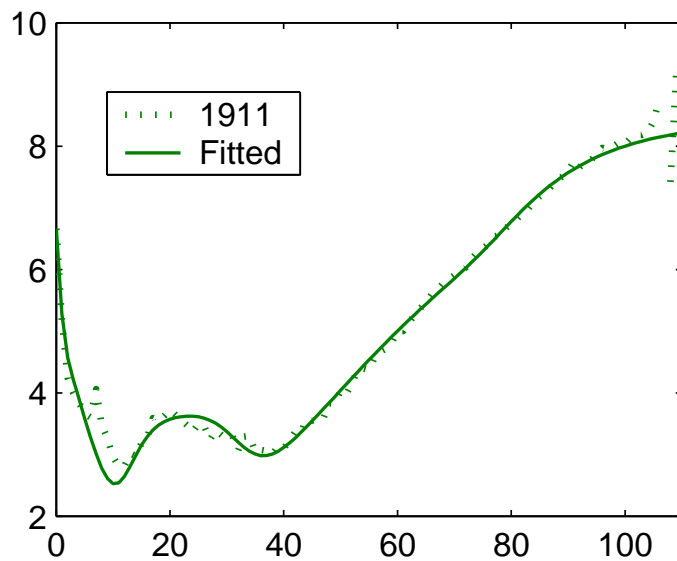
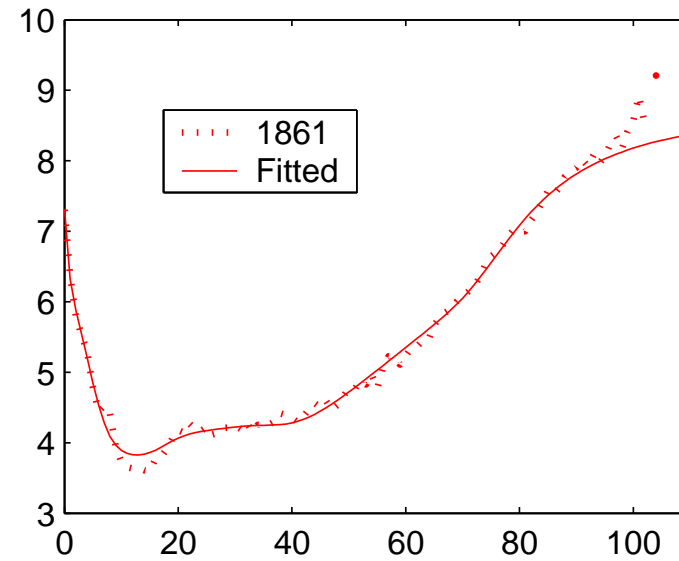
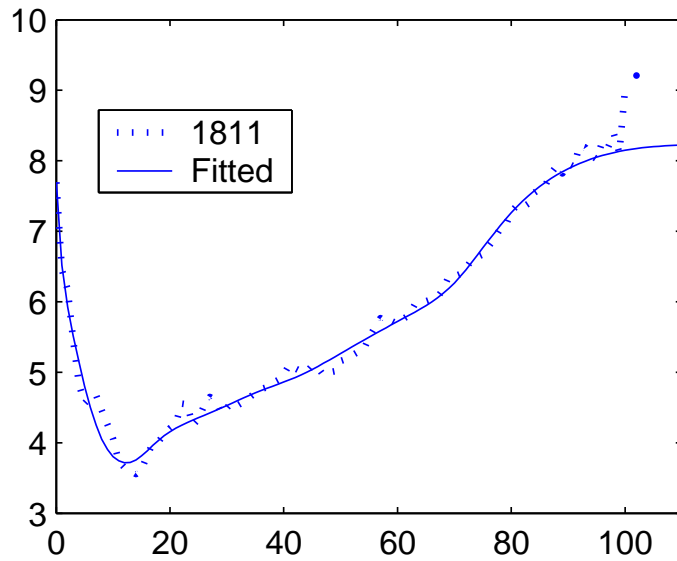


Estimated aging related parameters

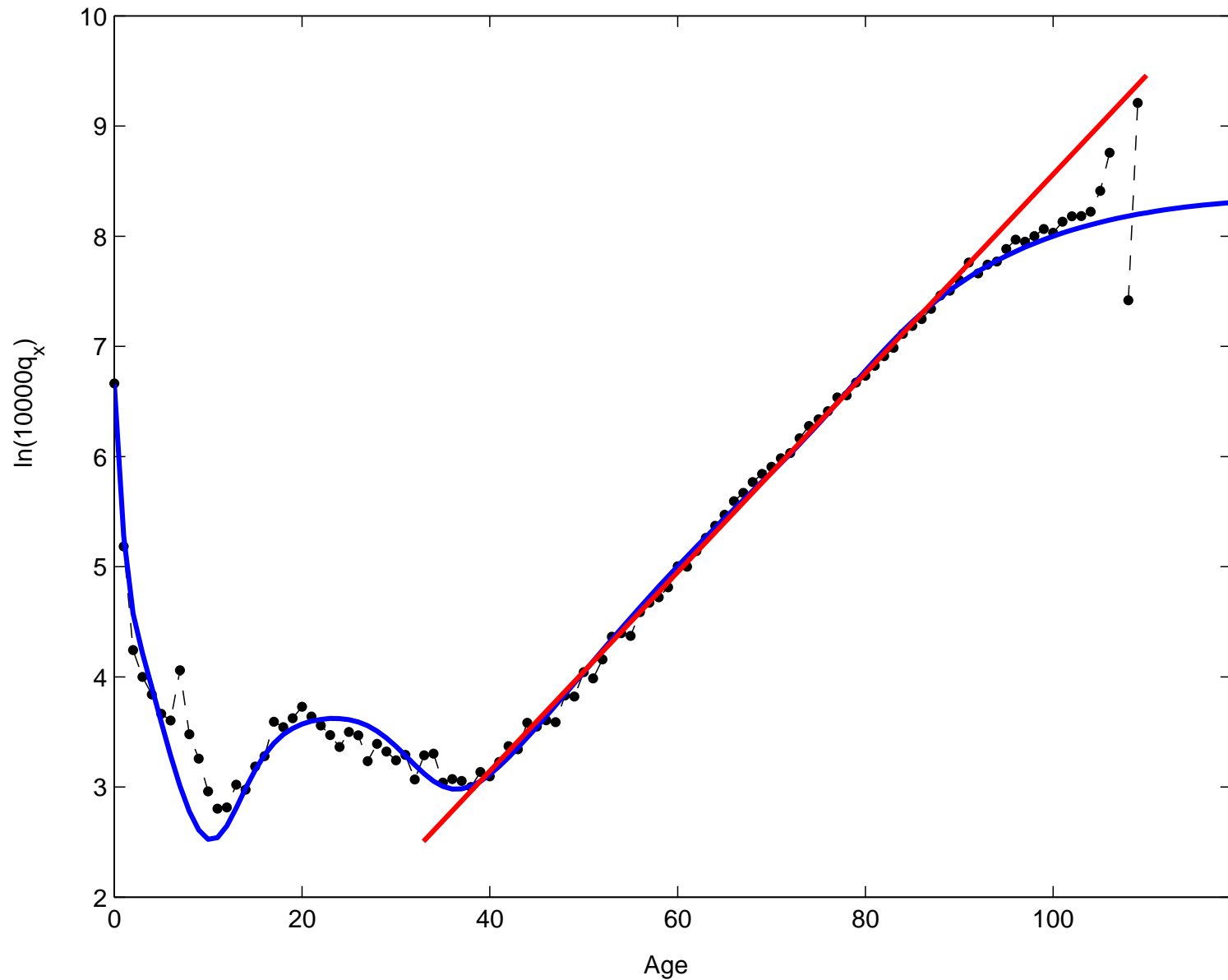
Table 1: Estimated aging related parameters for Swedish cohorts of year 1811, 1861, and 1911

Year	Parameters					
	λ	b	a	$[i_1, i_2]$	q	p
1811	2.5657	3.1504e-03	1.9888e-03	[42, 99]	9.3157e-09	3
1861	2.4794	4.4825e-03	1.9033e-03	[42, 89]	2.6351e-13	5
1911	2.3707	9.0987e-04	2.8939e-03	[33, 70]	1.8872e-15	6

Fitted curves on Sweden cohort 1811 to 1911



Fitted curve on Sweden cohort 1911



Conditional survival functions under PH mortality model

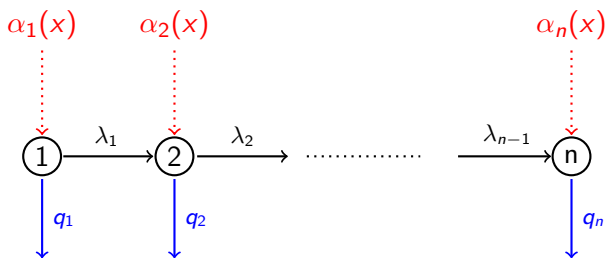
- ▶ Given that (x) survives to age $x + s$, the conditional probability that (x) will survive another t years is:

$$\begin{aligned}\frac{S_0(t+s)}{S_0(s)} &= \frac{\alpha e^{\Lambda(s+t)} \mathbf{e}}{\alpha e^{\Lambda s} \mathbf{e}} \\ &= \alpha_s e^{\Lambda t} \mathbf{e}, \quad t > 0,\end{aligned}$$

$$\text{where } \alpha_s = \frac{\alpha e^{\Lambda s}}{\alpha e^{\Lambda s} \mathbf{e}}.$$

- ▶ The future lifetime at time s again has a phase-type distribution, with a representation (α_s, Λ) of order n .
- ▶ Here α_s can be interpreted as the probability distribution of the cohort's health status at time s .
- ▶ α_s provides **added level of information** as a by-product of this aging process-based mortality model.

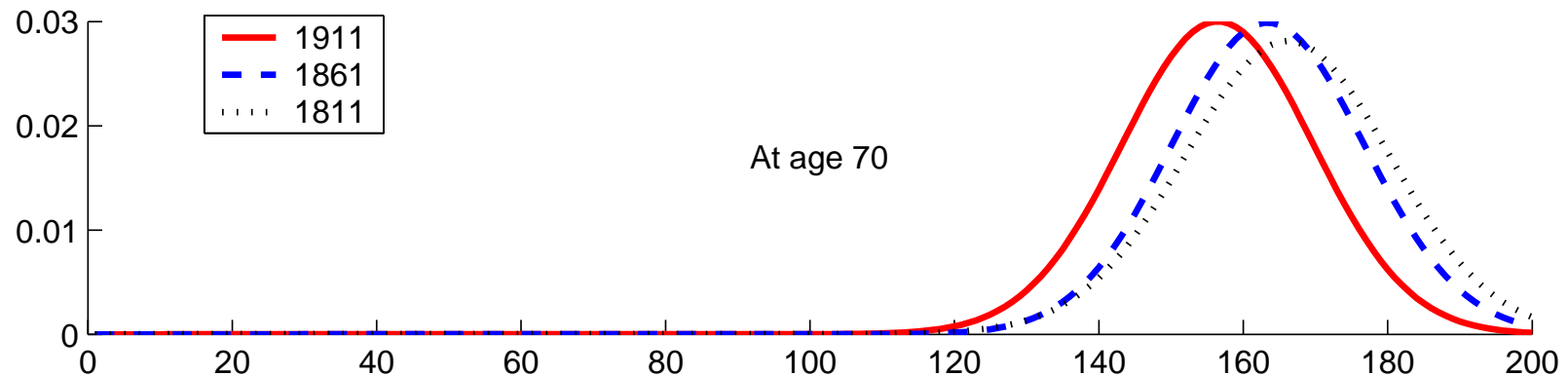
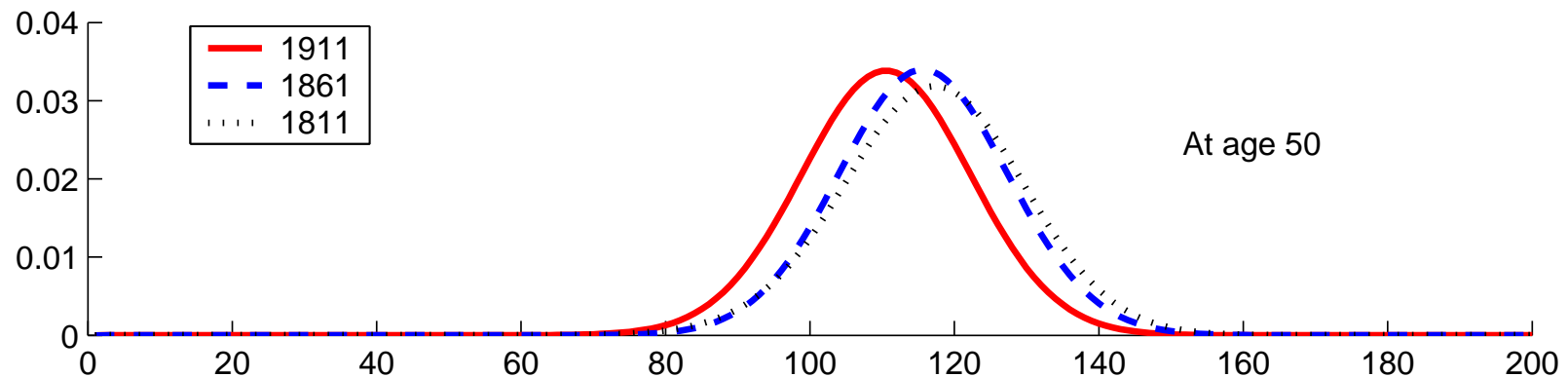
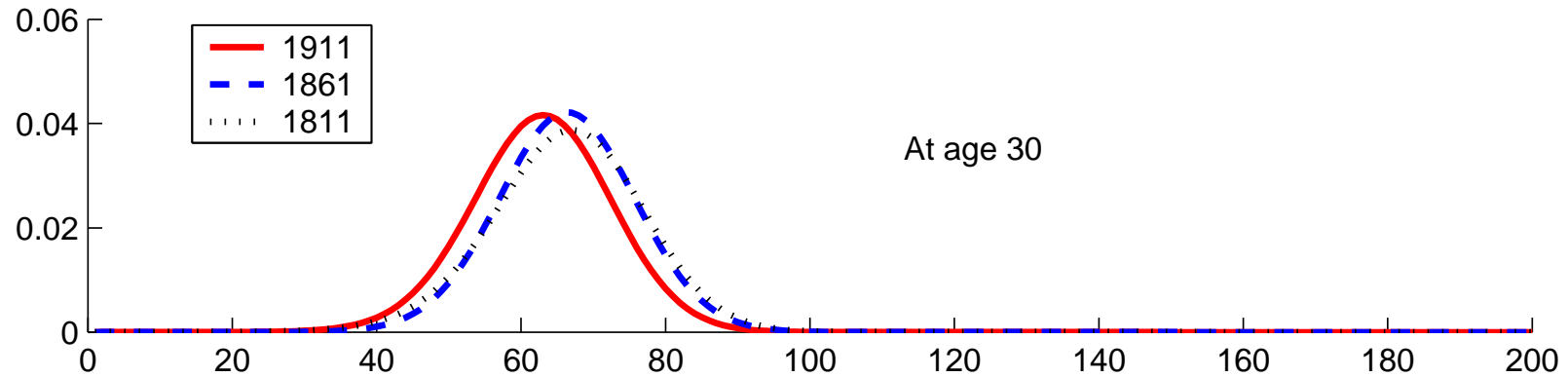
Physiological age distribution at age x



- ▶ At age x , the individual still follows the same aging process but with a different physiological age distribution, $\alpha(x)$, where

$$\alpha_i(x) = \left[\frac{\alpha e^{\Lambda x}}{\alpha \exp(\Lambda x) \mathbf{e}} \right]_i, \quad i = 1, \dots, n$$

Distributions on Physiological Ages of aged (x)



Mortality Uncertainty

We need to consider mortality uncertainty in prediction for pricing and risk management purposes.

Stochastic Mortality Modelling

Xiaoming Liu and X. Sheldon Lin (2012), A subordinated Markov model for stochastic mortality. *European Actuarial Journal* 2(1), 105-127.

- ▶ Introduce **mortality uncertainty** by subordinating Process.
- ▶ **Change measure** under the subordinated Markov modelling framework
- ▶ Calculate the **conditional expectation** as needed in the valuation of mortality-linked derivatives

Gamma Process and Subordinated Aging Process

- ▶ The gamma subordinating process:
 - ▶ $\gamma_0 = 0$;
 - ▶ it has independent increments, i.e., for any partition $0 \leq t_0 < t_1 < \dots < t_n$, the random variables $\gamma_{t_1} - \gamma_{t_0}, \dots, \gamma_{t_n} - \gamma_{t_{n-1}}$ are mutually independent; and
 - ▶ the increment $\gamma_{t+s} - \gamma_t$ has a gamma distribution with mean s and variance νs , for any $s, t \geq 0$.
- ▶ The aging process J_t is subordinated by the gamma process and the resulting aging process Z_t is now a subordinated Markov process

$$Z_t = J_{\gamma_t}.$$

Gamma Process and Subordinated Aging Process

- ▶ The gamma subordinating process:
 - ▶ $\gamma_0 = 0$;
 - ▶ it has independent increments, i.e., for any partition $0 \leq t_0 < t_1 < \dots < t_n$, the random variables $\gamma_{t_1} - \gamma_{t_0}, \dots, \gamma_{t_n} - \gamma_{t_{n-1}}$ are mutually independent; and
 - ▶ the increment $\gamma_{t+s} - \gamma_t$ has a gamma distribution with mean s and variance νs , for any $s, t \geq 0$.
- ▶ The aging process J_t is subordinated by the gamma process and the resulting aging process Z_t is now a subordinated Markov process

$$Z_t = J_{\gamma_t}.$$

- ▶ **Interpretation:** Allow aging process be altered by factors such as health care system change, environment change or medical science breakthrough, etc.

Survival Index

- ▶ With the model, we have

$$S(t) = S_0(\gamma_t) = \alpha e^{\mathbf{\Lambda}\gamma_t} \mathbf{e}, \quad t > 0.$$

- ▶ The new survival function $S(t)$ is a stochastic process and is referred to as the survival index for the cohort under consideration.
- ▶ The subordinated Markov mortality model is fully specified by $(\alpha, \mathbf{\Lambda}, \nu)$. The uncertainty degree is governed by parameter ν .

Term Structure of Mortality

- ▶ For $0 \leq s \leq t$, let $P(s, t)$ be the survival function of a life aged x at time 0 to be alive from time s to time t (valued at time s). $\{P(s, t); 0 \leq s \leq t\}$ is commonly referred to as the **term structure** of stochastic mortality; it used to be **conditional probs** in the deterministic setting.



$$P(s, t) = \frac{1}{S(s)} E[S(t) | \mathcal{F}_s],$$

where $\mathcal{F}_t, t \geq 0$, is the filtration generated by $S(t)$.

- ▶ We have shown

$$P(s, t) = \alpha_{\gamma_s} e^{\tilde{\Lambda}(t-s)} \mathbf{e}, \quad 0 \leq s \leq t.$$

As a special case, the term structure at time 0 is given by

$$P(0, t) = \alpha e^{\tilde{\Lambda}t} \mathbf{e}, \quad t \geq 0.$$

Explicit Expression of Term Structure of Mortality

Suppose that the eigenvalues $-\lambda_1, \dots, -\lambda_n$ of the intensity matrix $\mathbf{\Lambda}$ are distinct. Let $\mathbf{h}_1, \dots, \mathbf{h}_n$ and $\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_n$ be their corresponding right and left eigenvectors such that $\boldsymbol{\nu}_i \mathbf{h}_i = 1$. It is known that $\boldsymbol{\nu}_i \mathbf{h}_j = 0$, $i \neq j$, $i, j = 1, \dots, n$. Then, $P(s, t)$ has the phase-type representation $(\boldsymbol{\alpha}_{\gamma_s}, \tilde{\mathbf{\Lambda}})$, where

$$\boldsymbol{\alpha}_{\gamma_s} = \frac{\boldsymbol{\alpha} e^{\mathbf{\Lambda} \gamma_s}}{\boldsymbol{\alpha} e^{\mathbf{\Lambda} \gamma_s} \mathbf{e}},$$

and

$$\tilde{\mathbf{\Lambda}} = - \sum_{i=1}^n \tilde{\lambda}_i \mathbf{h}_i \boldsymbol{\nu}_i,$$

with $\tilde{\lambda}_i$ being given by

$$\tilde{\lambda}_i = \frac{1}{\nu} \ln(1 + \nu \lambda_i).$$

Variance of Survival Index

The variance of $S(t)$ is given by

$$\text{Var} [S(t)] = (\boldsymbol{\alpha} \otimes \boldsymbol{\alpha}) \left[e^{(\widetilde{\boldsymbol{\Lambda}} \oplus \boldsymbol{\Lambda}) t} - e^{(\widetilde{\boldsymbol{\Lambda}} \oplus \widetilde{\boldsymbol{\Lambda}}) t} \right] (\mathbf{e} \otimes \mathbf{e}).$$

Kronecker sum and product

- ▶ Denote $\mathbf{D} = \text{diag}(-\lambda_1, \dots, -\lambda_n)$, then

$$\mathbf{D} \oplus \mathbf{D} = \text{diag}(\mathbf{D} - \lambda_1 \mathbf{I}, \mathbf{D} - \lambda_2 \mathbf{I}, \dots, \mathbf{D} - \lambda_n \mathbf{I}),$$

the Kronecker sum of \mathbf{D} to itself, is diagonal with diagonal entries $-\zeta_k$, $k = 1, \dots, n^2$, where

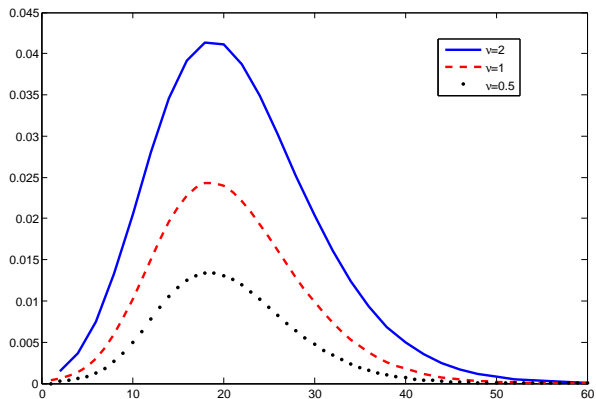
$$\zeta_{i+j} = \lambda_i + \lambda_j, \quad i, j = 1, \dots, n.$$

- ▶ Denote $\widetilde{\mathbf{D} \oplus \mathbf{D}} = \text{diag}(-\tilde{\zeta}_1, \dots, -\tilde{\zeta}_{n^2})$ and

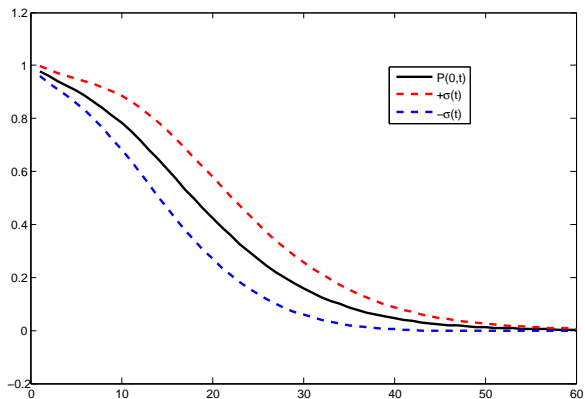
$$\widetilde{\mathbf{\Lambda} \oplus \mathbf{\Lambda}} = (\mathbf{H} \otimes \mathbf{H}) \left(\widetilde{\mathbf{D} \oplus \mathbf{D}} \right) (\mathbf{H} \otimes \mathbf{H})^{-1},$$

where $\mathbf{H} = (\mathbf{h}_1, \dots, \mathbf{h}_n)$ and \otimes is the symbol for the Kronecker product.

Variance function $\text{Var}[S(t)]$, $t \geq 0$, for $\nu = 0.5, 1$ and 2



Term structure $P(0, t)$ with one- σ confidence intervals,
based on $\nu = 1$



Interpretation of Parameter ν

- ▶ The variance function $\text{Var}[S(t)]$, $t \geq 0$, increases as ν gets larger.
- ▶ Hence, ν is the parameter to characterize longevity uncertainty.

Measure change under the PH model framework

Measure change under the PH model framework

- ▶ Under equivalent measure Q , define the intensity matrix

$$\Lambda_{\mathbf{u}} = \mathbf{u}\Lambda$$

where the matrix \mathbf{u} is a positive diagonal matrix:

$\mathbf{u} = \text{diag}(u_1, \dots, u_n)$, with $u_i > 0$, $i = 1, \dots, n$.

- ▶ $u_i, i = 1, \dots, n$, represent risk loadings with respect to the health statuses or frailty.
- ▶ Further, ν can be adjusted too. Let $\nu' = \nu + \nu_1$, here we call $\nu_1 (\geq 0)$ the longevity risk loading.
- ▶ Since the model structure is preserved, all calculations derived before are still applicable.

Difference between u change and ν change

$$\mathbf{\Lambda} = \begin{pmatrix} -0.2381937 & 0.2378846 & 0 & 0 & 0 \\ 0 & -0.2384834 & 0.237982 & 0 & 0 \\ 0 & 0 & -0.2390828 & 0.2380828 & 0 \\ 0 & 0 & 0 & -0.2423468 & 0.2413468 \\ 0 & 0 & 0 & 0 & -0.2433471 \end{pmatrix}$$
$$\tilde{\mathbf{\Lambda}} = \begin{pmatrix} -0.22829274 & 0.21863814 & 0.00882993 & 0.000475265 & 0.0000293618 \\ 0 & -0.22855900 & 0.21869466 & 0.00882899 & 0.000481546 \\ 0 & 0 & -0.22910983 & 0.21864396 & 0.00894398 \\ 0 & 0 & 0 & -0.23210733 & 0.221481270 \\ 0 & 0 & 0 & 0 & -0.23302529 \end{pmatrix}$$

Future works along this approach

- ▶ Improve fitting algorithm
- ▶ Identify connection with meaningful covariates
- ▶ develop mortality prediction method
- ▶ apply the model to insurance risk analysis, including pension and other mortality related products, such as reverse mortgage
- ▶ incorporate (Markov) interest rate model
- ▶ develop matrix analytic methodology for the related problems.

Applications in the literature based our approach

- ▶ Hautphenne and Latouche (2011), in the paper "The Markovian Binary Tree Applied to Demography" (Journal of Mathematical Biology, 2011, doi: 10.1007/s00285-011-0437-1), use our model framework in a comparison study of female population in different countries.
- ▶ Zadeh et al (2010), in the paper "The Use of Phase-type Models for Disability Insurance Calculations" (working paper submitted in refereed journal), explore the use of our proposed model framework in modeling disability rates and in the calculation of disability insurance.
- ▶ Su and Sherris (2011), in the paper "Heterogeneity of Australian Population Mortality and Implications for a Viable Life Annuity Market", Insurance: Mathematics and Economics 51:322-332, 2012, adopt our model in the study of population heterogeneity based on Australian population data.

Applications...

- ▶ Govorun et al (2011), in the paper "Profits and Risks of Pension Plans" (working paper presented on The Seventh International Conference on Matrix-Analytic Methods in Stochastic Models (MAM7), June 13-16, 2011), use our proposed model as a tool for profit estimation and risk management.
- ▶ Foo et al (2011), in the paper "Stochastic Dynamics of Cancer Initiation" (Physical Biology, 2011 (8), doi: 10.1088/1478-3975/8/1/015002), use a simplified version of our model for the lifetime variable of a cancer patient.