Pricing and Hedging in Incomplete markets with Model Ambiguity

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Outline

Literature Overview

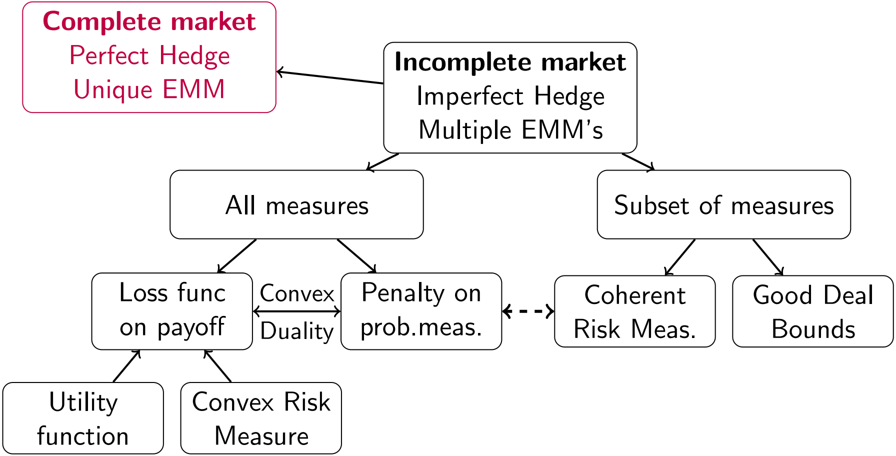


Figure: Overview of Pricing in Incomplete Markets

Complete Market (1)

Explain concepts on one-step binomial "fork":

$$
\begin{array}{cccl}
 & & uS & f_u \text{ with prob } p \\
S & \begin{array}{c}\nearrow \\ \searrow\end{array} & & \\
 & & dS & f_d \text{ with prob } 1-p \\
\hline
B & \rightarrow & RB & \text{certain}
\end{array}
$$

At $t = 0$ buy replicating portfolio $\Delta S + B$ to replicate uncertain payoff (f_u, f_d) at $t = 1$.

Complete Market (2)

Solve system of equations to match payoff:

$$\begin{cases} u\Delta S + RB = f_u \\ d\Delta S + RB = f_d \end{cases}$$

Solution given by:

$$\Delta S = \frac{f_u - f_d}{u - d}, \quad B = \frac{uf_d - df_u}{R(u-d)}$$

Market price of replio at $t = 0$:

$$\Delta S + B = \frac{(R-d)f_u + (u-R)f_d}{R(u-d)} = \frac{1}{R}\big(qf_u + (1-q)f_d\big)$$

q well-defined iff $u > R > d$ (FTAP: [Delbaen and Schachermayer, 1994])
Hence: "risk-neutral" pricing is convenient representation of the market price of the replio

Binomial Incomplete Market

Note: in real world we *never* have complete market!

Model for incomplete market:
Combine two binomial forks: financial (u, d) and insurance (g, b)

$$
\begin{array}{ccccc}
 & & & \nearrow & f_{ug} \text{ with prob } p\pi \\
 & \nearrow & uS & \searrow & f_{ub} \text{ with prob } p(1-\pi) \\
S & & & \nearrow & f_{dg} \text{ with prob } (1-p)\pi \\
 & \searrow & dS & \searrow & f_{db} \text{ with prob } (1-p)(1-\pi) \\
\hline
B & & \longrightarrow & & RB
\end{array}
$$

For simplicity we use independent prob's
Can only trade (S, B). Insurance event (g, b) makes market incomplete.
How do we price the uncertain payoff f?

Market-Consistent & Time-Consistent (1)

For pricing insurance and pension contracts we want to consider realistic pricing operators $v[t, X]$ that are:

- market consistent
- time consistent

Market-consistent (MC): $v[t, X + H] = v[t, X] + \mathbb{E}^{\mathbb{Q}}[H \mid \mathcal{F}_t]$ for all hedgeable claims H

This is an extension of "cash invariance" to all H

Time-consistent(TC): if $X_1 \geq X_2$ at time T, then $v[t, X_1] \geq v[t, X_2]$ for all $t < T$

This is an extension of no-arbitrage to incomplete markets.

Note: the conditional expectation operator $\mathbb{E}^{\mathbb{Q}}[H \mid \mathcal{F}_t]$ is TC (and MC)

Market-Consistent & Time-Consistent (2)

Result by [Pelsser and Stadje, 2014]: every MC and TC pricing operator can be represented as a "two-step" operator:

$$
\begin{array}{ccccc}
 & & & \nearrow & f_{ug} \\
 & \nearrow & f_u := v[f \mid uS] & & \\
 & & & \searrow & f_{ub} \\
\frac{1}{R}\big(qf_u + (1-q)f_d\big) & & & & \\
 & & & \nearrow & f_{dg} \\
 & \searrow & f_d := v[f \mid dS] & & \\
 & & & \searrow & f_{db}
\end{array}
$$

"financial" valuation "actuarial" valuation

Clean separation of financial and actuarial pricing in each "half-step".

Tech. cond: financial info arrives more frequently than insurance info

Binomial Insurance Risk

We will now concentrate on the "actuarial" pricing step, in a "pure" incomplete market

One-step binomial "fork" for insurance process y:

$$
y \begin{array}{l} \nearrow \quad y + a\Delta t + b\sqrt{\Delta t} \quad \text{with prob } \frac{1}{2} \\ \\ \searrow \quad y + a\Delta t - b\sqrt{\Delta t} \quad \text{with prob } \frac{1}{2} \end{array}
$$

The term $\pm\sqrt{\Delta t}$ is approximation of Brownian Motion increment ΔW

In the limit $\Delta t \to 0$ this converges to the sde $dy = a(t,y)dt + b(t,y)dW$

Binomial Insurance Risk – Girsanov

Consider the change of probability measure:

$$
\begin{array}{ccl}
 & \nearrow & y + a\Delta t + b\sqrt{\Delta t} \quad \text{with prob } \frac{1}{2}(1+\lambda\sqrt{\Delta t}) \\
y & & \\
 & \searrow & y + a\Delta t - b\sqrt{\Delta t} \quad \text{with prob } \frac{1}{2}(1-\lambda\sqrt{\Delta t})
\end{array}
$$

Factor $(1+\lambda\Delta W)$ is the Radon-Nikodym derivative

Tree above is equivalent to process y with adjusted drift:

$$
\begin{array}{ccl}
 & \nearrow & y + (a+\lambda b)\Delta t + b\sqrt{\Delta t} \quad \text{with prob } \frac{1}{2} \\
y & & \\
 & \searrow & y + (a+\lambda b)\Delta t - b\sqrt{\Delta t} \quad \text{with prob } \frac{1}{2}
\end{array}
$$

Hence: $\mathbb{E}_t^{(\lambda)}[y_{t+\Delta t}] = y_t + (a+\lambda b)\Delta t = \mathbb{E}_t[y_{t+\Delta t}] + \lambda b\Delta t$.

Actuarial Variance and Utility Pricing

Variance pricing operator: $v_{\text{var}}(t, y) := \mathbb{E}_t[y_{t+\Delta t}] + \alpha \mathbb{V}\text{ar}_t[y_{t+\Delta t}]$
In our binomial model: $v_{\text{var}}(t, y) = \mathbb{E}_t[y_{t+\Delta t}] + \alpha b^2 \Delta t$
But, this is equivalent to: $\mathbb{E}_t^{(\lambda)}[y_{t+\Delta t}]$ with $\lambda = \alpha b$
Reference: [Kaas et al., 2008]

Utility pricing operator: $v_{\text{util}}(t, y) := u^{-1}\left(\mathbb{E}_t[u(y_{t+\Delta t})]\right)$
In our binomial model: $v_{\text{util}}(t, y) \approx \mathbb{E}_t[y_{t+\Delta t}] + \frac{1}{2}\frac{u''}{u'} b^2 \Delta t$
But, this is equivalent to: $\mathbb{E}_t^{(\lambda)}[y_{t+\Delta t}]$ with $\lambda = \frac{1}{2}\frac{u''}{u'} b$
Note: $\lambda(t, y)$ can be state dependent, e.g. with $u(t, y)$
References: [Musiela and Zariphopoulou, 2004], [Carmona, 2009]

Cost-of-Capital Pricing

Prescribed in the standard model of Solvency II
$v_{\mathrm{CoC}}(t, y) := \mathbb{E}_t[y_{t+\Delta t}] + \gamma\sqrt{\Delta t}\mathrm{VaR}_t[y_{t+\Delta t}]$
Note: standard model uses time-step of $\Delta t = 1$ year
We include factor $\sqrt{\Delta t}$ to allow scaling to different time horizons
For small time-step $\Delta t \to 0$ the VaR[] $\to k\sqrt{\mathbb{V}\mathrm{ar}[]}$
Standard model uses 99.5% prob, hence $k = \Phi^{-1}(0.995) = 2.58$

In our binomial model: $v_{\mathrm{CoC}}(t, y) = \mathbb{E}_t[y_{t+\Delta t}] + \gamma k|b|\Delta t$
But, this is equivalent to: $\mathbb{E}_t^{(\lambda)}[y_{t+\Delta t}]$ with $\lambda = \gamma k \mathrm{sign}(b)$
Standard model corresponds to: $\lambda = 0.06 * 2.58 = 0.15$.

Coherent and Convex Risk Measures

Axiomatic foundation of risk measure / pricing operator (note $\pm$-switch)
Started with coherent: [Artzner et al., 1999]
Extended to convex: [Föllmer and Schied, 2002]
Extension to time-consistent convex: [Cheridito et al., 2006]

Any TC convex pricing operator can be characterised as:
(In our binomial model:) $v_{\text{cvx}}(t,y) = \max_\lambda \mathbb{E}_t^{(\lambda)}[y_{t+\Delta t}] - c(\lambda)\Delta t$
convex penalty function $c(\lambda) \geq 0$ and $c(0) = 0$
Equiv to: $v_{\text{cvx}}(t,y) = \mathbb{E}_t[y_{t+\Delta t}] + \big(\max_\lambda \lambda b - c(\lambda)\big)\Delta t$
Equiv to: $v_{\text{cvx}}(t,y) = \mathbb{E}_t[y_{t+\Delta t}] + \tilde{c}(b)\Delta t$
where $\tilde{c}(b)$ is the *convex dual* of $c(\lambda)$

Interpretation: pricing based on “worst case” measure λ, but with a penalty for choosing “too large” values of λ.

Coherent Risk Measures

Special case of convex risk measure, with penalty function:

$$c(\lambda) = \begin{cases} 0 & \text{for } |\lambda| \leq k; \\ \infty & \text{for } |\lambda| > k. \end{cases}$$

Convex dual is given by: $\tilde{c}(b) = \max_{|\lambda| \leq k} \lambda b$
LP with explicit solution: $\tilde{c}(b) = k|b|$

Hence: $v_{\text{coh}}(t, y) = \mathbb{E}_t[y_{t+\Delta t}] + k|b|\Delta t$
Note: Cost-of-Capital is a coherent pricing operator.

Good Deal Bound

Consider the change of probability measure:

$$
y \begin{array}{l} \nearrow \quad y + a\Delta t + b\sqrt{\Delta t} \quad \text{with prob } \frac{1}{2}(1+\lambda\sqrt{\Delta t}) \\ \\ \searrow \quad y + a\Delta t - b\sqrt{\Delta t} \quad \text{with prob } \frac{1}{2}(1-\lambda\sqrt{\Delta t}) \end{array}
$$

We can also interpret λ as *market price of risk* (MPR)

Arbitrage opportunity is "ridiculous good deal" with $\lambda \to \pm\infty$

Good deal bound: only look at measures with MPR not too high
Hence: only consider measures with $|\lambda| \leq k$
Hence: $v_{\text{GDB}}(t, y) = v_{\text{coh}}(t, y) = \mathbb{E}_t[y_{t+\Delta t}] + k|b|\Delta t$

References: [Cochrane and Saá-Requejo, 2000], [Björk and Slinko, 2006]

Literature Overview – Recap

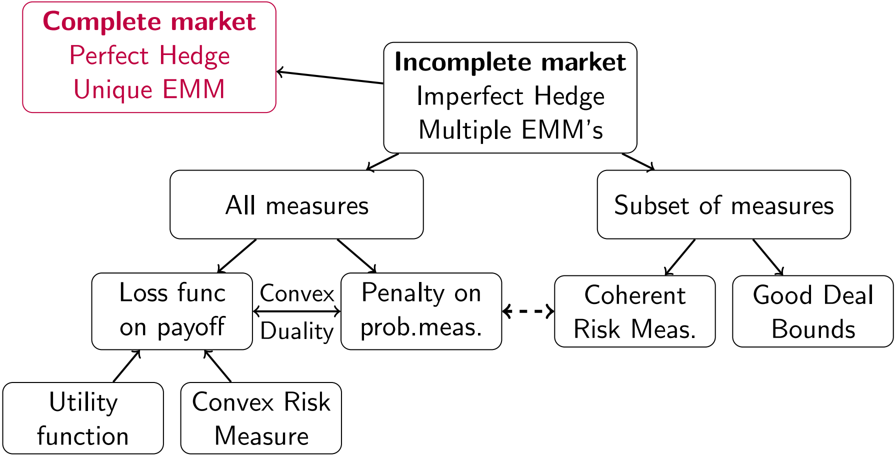


Figure: Overview of Pricing in Incomplete Markets

Pricing and Hedging with Model Ambiguity

Ambition: "general theory" for pricing and hedging in incomplete markets

Minimal # of assumptions:

1. Agent (or company) wants to maximise surplus
2. Agent (or company) is able to trade financial risk
3. Concerned about model mis-specification

Model Ambiguity

How can we handle model mis-specification mathematically?
Consider models near "central model" $dy = a(t,y)dt + b(t,y)dW$:

$$
y \begin{array}{l} \nearrow \quad y + a\Delta t + b\sqrt{\Delta t} \quad \text{with prob } \frac{1}{2}(1+\lambda\sqrt{\Delta t}) \\ \\ \searrow \quad y + a\Delta t - b\sqrt{\Delta t} \quad \text{with prob } \frac{1}{2}(1-\lambda\sqrt{\Delta t}) \end{array}
$$

Models near "central model" correspond to small values of λ

We consider the set of models $|\lambda| \leq k$, where k is chosen based on estimation error or statistical test procedure.
Note: stochastic $\lambda(t,y)$ give a very rich class of alternative models.

Intuition (2)

1. Objective of agent

$$\max_{\bar{\boldsymbol{\theta}}(t)} \mathbb{E}[A(T, \bar{\boldsymbol{x}}_T, \bar{\boldsymbol{\theta}}_T) - L(T, \bar{\boldsymbol{x}}_T, \boldsymbol{y}_T)]$$

2. Set of alternative models

$$\mathcal{K} = \left\{\boldsymbol{\mu}(t, \boldsymbol{z}) + \boldsymbol{\epsilon}(t) | \boldsymbol{\epsilon}(t)' \boldsymbol{\Sigma}^{-1}(t) \boldsymbol{\epsilon}(t) \leq k^2\right\}$$

3. Robust optimisation specification via "two-player game":

$$\max_{\bar{\theta}(t)} \min_{\mathbb{Q} \in \mathcal{K}} \mathbb{E}^{\mathbb{Q}}[A(T, \bar{\boldsymbol{x}}_T, \bar{\boldsymbol{\theta}}_T) - L(T, \bar{\boldsymbol{x}}_T, \boldsymbol{y}_T) | \mathcal{F}_t]$$

Model

- n tradeable assets $\boldsymbol{x}(t) = \{x_i(t) | i = \{1, ..., n\}\}$
- k untradeable assets $\boldsymbol{y}(t) = \{y_i(t) | i = \{1, ..., k\}\}$
- Bank account $dx_0 = rx_0 dt$

$$d \begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{y}(t) \end{pmatrix} = \begin{pmatrix} \mu_{\boldsymbol{x}}(t, \boldsymbol{x}) \\ \mu_{\boldsymbol{y}}(t, \boldsymbol{y}) \end{pmatrix} dt + \boldsymbol{\Sigma}^{1/2}(t) \begin{pmatrix} d\boldsymbol{W}_{\boldsymbol{x}}(t) \\ d\boldsymbol{W}_{\boldsymbol{y}}(t) \end{pmatrix}$$

Model (2)

- $\bar{\boldsymbol{\theta}}(t) = [\overbrace{\theta_0(t),}^{1}\overbrace{\theta_1(t), ..., \theta_n(t)}^{n}\overbrace{0, ..., 0}^{k}]$
- Hedging portfolio $A(t, \boldsymbol{x}_t)$
- Liability $L(t, \boldsymbol{x}_t, \boldsymbol{y}_t)$
- Optimisation by Hamilton-Jacobi-Bellman $\max_{\boldsymbol{\theta}(t)} \min_{\epsilon(t)} \mathbb{E}[A(T) - L(T, \boldsymbol{z}_T)|\mathcal{F}_t] = v(t, A, \boldsymbol{x}, \boldsymbol{y})$
- Indifference pricing:
    - No contract $v(t, A, \boldsymbol{x}, \boldsymbol{y}) = A(t)e^{r(T-t)}$
    - Extra cash needed to make agent indifferent $v(t, A, \boldsymbol{x}, \boldsymbol{y}) = (A(t) + \pi(t, \boldsymbol{x}, \boldsymbol{y}))e^{r(T-t)} - w(t, \boldsymbol{x}, \boldsymbol{y})$
    - Solves for $\pi(t, \boldsymbol{x}, \boldsymbol{y}) = e^{-r(T-t)}w(t, \boldsymbol{x}, \boldsymbol{y})$
- Represent price $\pi()$ as semi-linear PDE

Theorem (Intuitive representation of Theorem)

The indifference price π is given by the PDE

$$\pi_t + \mathcal{L}_{\mu,\sigma}\pi - r\pi + c\sqrt{\boldsymbol{\pi}_{\boldsymbol{y}}'\boldsymbol{S}\boldsymbol{\pi}_{\boldsymbol{y}}} = 0$$

where $\mathcal{L}_{\mu,\sigma}\pi$ is the Feynman-Kaç part, with terminal value $\pi(T, \boldsymbol{x}, \boldsymbol{y}) = L(T, \boldsymbol{x}, \boldsymbol{y})$ and where $c = \sqrt{k^2 - \boldsymbol{q}_{\boldsymbol{x}}{}'\boldsymbol{\Sigma}_{\boldsymbol{xx}}{}^{-1}\boldsymbol{q}_{\boldsymbol{x}}}$

The term $c\sqrt{\boldsymbol{\pi}_{\boldsymbol{y}}'\boldsymbol{S}\boldsymbol{\pi}_{\boldsymbol{y}}}$ is the k-dimensional generalisation of "$k|b|$".

Lemma (Optimal hedging portfolio and robustness adjustments)

The agent's optimal dynamic hedging portfolio is

$$\boldsymbol{\theta}^* = \begin{bmatrix} \boldsymbol{\pi_x} + \boldsymbol{\Sigma}_{\boldsymbol{xx}}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{xy}}\boldsymbol{\pi_y} + h\boldsymbol{\Sigma}_{\boldsymbol{xx}}^{-1}\boldsymbol{q_x} \\ \mathbf{0} \end{bmatrix}$$

Mother nature's optimal drift adjustments are

$$\boldsymbol{\epsilon}^* = \begin{bmatrix} -\boldsymbol{q_x} \\ \boldsymbol{S}h^{-1}\boldsymbol{\pi_y} - \boldsymbol{\Sigma_{xy}}'\boldsymbol{\Sigma_{xx}}^{-1}\boldsymbol{q_x} \end{bmatrix}$$

where $h = \sqrt{\frac{\boldsymbol{\pi_y}'\boldsymbol{S}\boldsymbol{\pi_y}}{k^2 - \boldsymbol{q_x}'\boldsymbol{\Sigma_{xx}}^{-1}\boldsymbol{q_x}}}$.

Example: Pure hedgeable risk

- Let $k = 0$
- Complete market

$$\pi_t + \mathcal{L}_{\mu,\sigma}\pi - r\pi + 0 = 0$$

- Risk-neutral pricing

Example: Uncorrelated non-traded asset

- Liability function $L(T, S, N) = \max(S_T, g) N_T$
- Unit-linked contract with N_T survivors
- The stochastic processes are

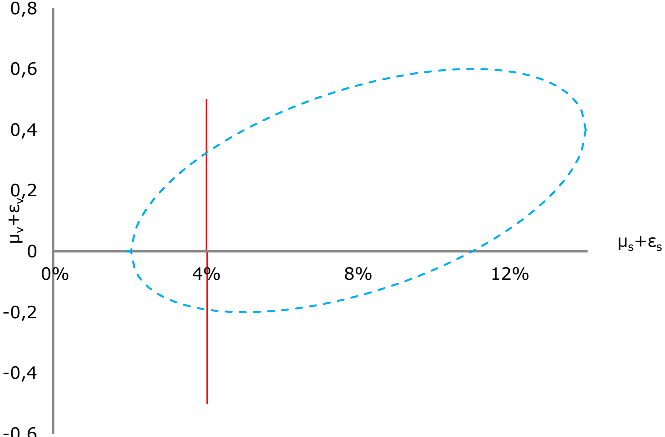
$$d \begin{bmatrix} S_t \\ N_t \end{bmatrix} = \begin{bmatrix} \mu S_t \\ -\alpha N_t \end{bmatrix} dt + \begin{bmatrix} \sigma S_t & 0 \\ 0 & \beta N_t \end{bmatrix} \begin{bmatrix} dW_S \\ dW_N \end{bmatrix}$$

- Price in our model:

$$\begin{aligned} \pi(t, S_t, N_t) &= \left(S_t \Phi(d_1) - e^{-r(T-t)} g \Phi(d_2) + e^{-r(T-t)} g \right) \times \\ & N_t e^{\left(-\alpha + \beta \sqrt{k^2 - ((\mu - r)/\sigma)^2}\right)(T-t)} \end{aligned}$$

- This is actuarial prudence with adjusted drift $-\alpha + \beta\sqrt{\cdots}$

Graphical Representation of Model Ambiguity



Conclusion

- Based on minimal assumptions
  - Agent maximising surplus
  - Uncertain about model
  - Able to hedge
- Complete market $\Rightarrow$ risk-neutral measure
- Incomplete $\Rightarrow$ pricing with actuarial prudence
- Hedgeable $+$ unhedgeable risk $\Rightarrow$ MC and TC pricing and hedging
- Multivariate $\Rightarrow$ unique solution

References I

Artzner, P., Delbaen, F., Eber, J., and Heath, D. (1999).
Coherent measures of risk.
Mathematical Finance, 9(3):203–228.

Björk, T. and Slinko, I. (2006).
Towards a general theory of good-deal bounds.
Review of Finance, 10(2):221–260.

Carmona, R. (2009).
Indifference pricing: theory and applications.
Princeton University Press.

Cheridito, P., Delbaen, F., and Kupper, M. (2006).
Dynamic monetary risk measures for bounded discrete-time processes.
Electronic Journal of Probability, 11(3):57–106.

References II

Cochrane, J. H. and Saá-Requejo, J. (2000).
Beyond arbitrage: good-deal asset price bounds in incomplete markets.
Journal of Political Economy, 108(1):79–119.

Delbaen, F. and Schachermayer, W. (1994).
A general version of the fundamental theorem of asset pricing.
Mathematische Annalen, 300(3):463–520.

Föllmer, H. and Schied, A. (2002).
Convex measures of risk and trading constraints.
Finance and Stochastics, 6(4):429–447.

Kaas, R., Goovaerts, M., Dhaene, J., and Denuit, M. (2008).
Modern Actuarial Risk Theory: Using R.
Springer Verlag.

References III

Musiela, M. and Zariphopoulou, T. (2004).
An example of indifference prices under exponential preferences.
Finance and Stochastics, 8(2):229–239.

Pelsser, A. and Stadje, M. (2014).
Time-consistent and market-consistent evaluations.
Mathematical Finance, 24(1):25–65.