SOLVENCY MEASURE FOR PENSION LIABILITIES : time, inflation and longevity

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PRM

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PENSION

+

ERM

The Solvency Issue

INSURANCE → SOLVENCY 2

BANKS → BASEL 2 / 3

PENSION FUNDS → ???

IAS Norms : Best estimate approach ; not risk based !

Outline

- 1. Pension and solvency
- 2. Market risk Model
- 3. Risk Measure Approach
- 4. Probability of ruin Approach
- 5. Introduction of the Longevity Risk
- 6. Extension to DB schemes

1. Pension and solvency

Risk mapping of pension liabilities :

- MARKET RISKS :
  - discount in the Actuarial Liabilities
  - return on the Assets

- ECONOMICAL RISKS :
  - inflation and salary increase

- DEMOGRAPHIC RISKS :
  - longevity risk
  - withdrawal risk

+ Long term aspect !!!

DB versus DC Pension schemes :

In **DC Schemes** (*Defined Contribution Pension Schemes where the only liability of the sponsor is to pay a well defined level of contribution*), risks are essentially transferred to the affiliates.

Market risk can be present in case of guarantee of a minimum rate of return on the contributions .

In **DB Schemes** (*Defined Benefit Pension Schemes where the sponsor promises a well defined level of benefit at retirement*), risks are clearly the problem of the sponsor !!

DC model

We consider a ***pension investment product*** (DC plan) with a fixed minimum guaranteed return given at retirement.

Notations :

P=1 :	initial pension account at time t=0
N :	maturity of the pension account
i :	fixed guaranteed return
r :	risk free rate
L(N) :	liability at retirement age

0 N

$$P = 1$$

$$L(N) = (1+i)^N = e^{r_G N}$$

On the asset side we assume the insurer invests the initial contribution in a risky asset.

A(t) = asset at time t (*stochastic evolution*)

A(N) is a random variable …

3 fundamental issues in ***DC / Funding*** for the affiliates :

1°) the optimal investment strategy of the contributions : *risk / return*

2°) An eventual minimum guaranteed return

3°) the payment of the benefits : *lump sum or annuity*

Problem 1 : Solvency at maturity :

Ability of the fund to fulfill the guarantee only *at maturity* and *without any additional equity*

Probability of no default at maturity :

Assets must be enough only at maturity to match the liability :

$$P(A(N) \geq L(N))$$

Problem 2 : Risk measure at maturity :

Ability of the fund to fulfill the pension guarantee only *at maturity* ***with eventual additional equity***

EXAMPLE : Static value at risk :

Initial amount C to put in the fund in order to guarantee with a high probability the liability at maturity :

$$P(A(N) + C e^{rN} \geq L(N)) \geq \alpha \quad (99\%)$$

Problem 3 : Ruin theory approach :

Ability of the fund to fulfill the guarantee ***at any time*** *with eventual additional equity* (provision approach)

EXAMPLE : Probability of ruin :

Reserve process :

$$\boxed{R(s) = A(s) + C.e^{rs} - L(s)}$$

$$\text{for}: \quad 0 \le s \le N$$

$$\text{with}: \; A(s) = \text{market value of Assets at time } s$$

$$C = \text{initial solvency Capital at time } 0$$

$$L(s) = \text{value of Liabilities at time } s \;\; (\; ?)$$

Problem 3 : Ruin theory approach (2) :

Initial own funds C to put in the pension fund in order to guarantee with a high probability the solvency at any time before maturity :

$$\phi(C) = P(R(s) \geq 0,\ \forall s \in [0, N])$$

The initial capital C is then chosen such that :

$$\boxed{\phi(C) \geq \alpha}$$

2. Market Risk Model

Assumptions :

x=30 x=50 x=65

time t

Existing amount
at time t=0
S(0)=1

t=N
Maturity
(retirement age)

S (t, N) = value of the asset at time t for a maturity N

Model :

Risky asset :

$$dX(t) = \delta.X(t)\,dt + \sigma.X(t)\,dw(t)$$

mean return *volatility*

Riskless asset :

$$dB(t) = r.B(t)\,dt$$

riskfree rate

Mixing :

$$\alpha(t,N) = \text{part invested at time } t \text{ in the risky asset for a maturity } N \text{ (measurable process)}$$

Stochastic differential equation of the mixed asset :

$$dS(t,N) = (\alpha(t,N).\delta + (1-\alpha(t,N)).r).\,S(t,N)\,dt + \alpha(t,N).\sigma.S(t,N)\,dw(t)$$

The solution of this equation is given by :

$$S(t) = S(0).\exp((r.t + (\delta - r)\int_0^t \alpha(s)ds - \sigma^2/2\int_0^t \alpha^2(s)ds + \sigma\int_0^t \alpha(s)\,dw(s))$$

Investment strategy = how to choose the process α(t) ?

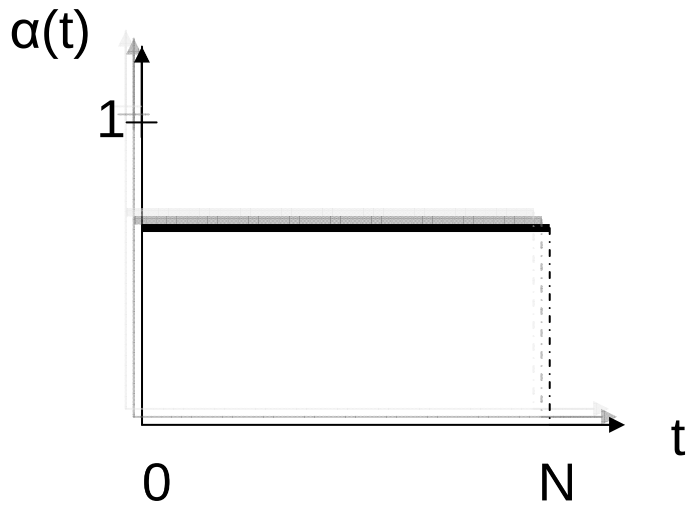
Mixed strategy

We consider 3 deterministic investment strategies mixing :

- a risky asset (stocks, options,…)
- a riskless asset (bonds)

Strategy 1 : **Constant proportion portfolio**
Strategy 2 : **Linear decreasing strategy**
Strategy 3 : **Lifecycle strategy**

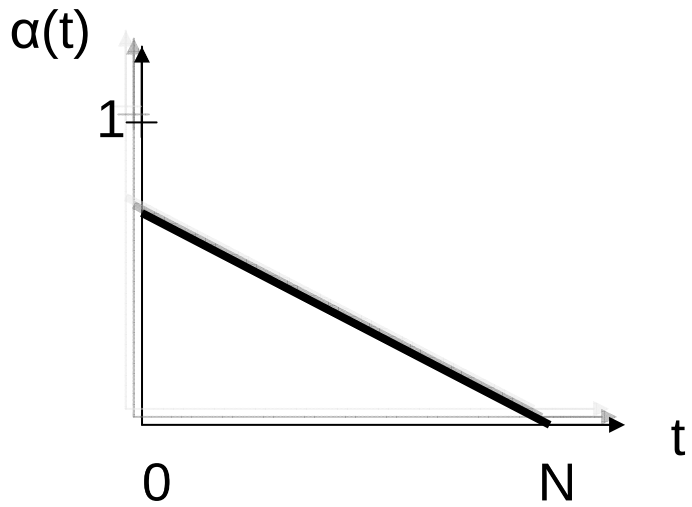
Constant proportion portfolio



Fixed constant proportion in the risky asset during all the horizon
(MERTON : *optimal strategy using utility function arguments*)

1 parameter : - constant proportion

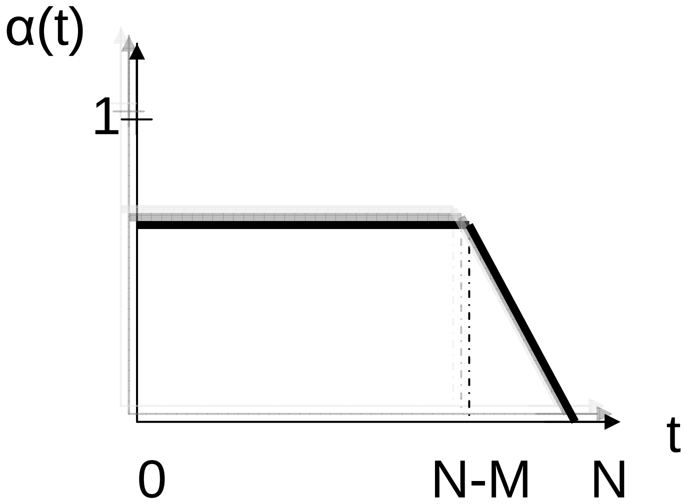
Linear decreasing strategy



Linear decreasing proportion in the risky asset during all the horizon

1 parameter : - initial proportion

Lifecycle strategy



Fixed constant proportion in the risky asset during a first period (*accumulation phase*) and then linearly decreasing until retirement (*consolidation phase*)

2 parameters : - initial proportion
- consolidation period M

Constant proportion portfolio

$$\boxed{\alpha(t) = \gamma} \quad \forall 0 \leq t \leq N$$

For instance : γ= 50%

Solution :

$$\boxed{S(t,N) = S(0).\exp((r + \gamma.(\delta - r))t - \gamma^2\sigma^2 t/2 + \gamma\sigma.w(t))}$$

Independent of the maturity for a fixed time t !

Mean : $$\boxed{ES(t,N) = S(0).\exp((r + \gamma.(\delta - r))t)}$$

Constant proportion portfolio

Amount at retirement age (maturity t=N) :

$$S(N,N) = S(0).\exp\{N(m(N), V^2(N))\}$$

with :

$$m(N) = r.N + \gamma.(\delta - r).N - \sigma^2\gamma^2 N/2$$
$$V^2(N) = \sigma^2\gamma^2 N$$

Mean :

$$ES(N,N) = S(0).\exp(r\,N + \gamma(\delta - r)N)$$

Linear decreasing strategy

$$\boxed{\alpha(t) = \beta.(1 - \frac{t}{N})} \quad \forall 0 \le t \le N$$

For instance : N = 20

β = 70%

$$\alpha(0) = 70\%$$
$$\alpha(5) = 52.5\%$$
$$\alpha(10) = 35\%$$
$$\alpha(15) = 17.5\%$$
$$\alpha(20) = 0\%$$

Linear decreasing strategy

Amount at retirement age (t=N) :

$$S(N,N) = S(0).\exp\{N(m(N), V^2(N))\}$$

with :

$$m(N) = r.N + \beta.(\delta - r).N/2 - \sigma^2\beta^2 N/6$$
$$V^2(N) = \sigma^2\beta^2 N/3$$

Mean :

$$ES(N,N) = S(0).\exp(r\,N + \beta(\delta - r)N/2)$$

Lifecycle strategy

$$\alpha(t) = \alpha \qquad \text{for } 0 \le t \le N - M$$

$$\frac{\alpha}{M}(N - t) \qquad \text{for } N - M \le t \le N$$

For instance : N = 30
M = 15
α = 70%

$\alpha(0) = 70\%$ $\alpha(5) = 70\%$

$\alpha(10) = 70\%$ $\alpha(15) = 70\%$

$\alpha(20) = 47\%$ $\alpha(25) = 23\%$ $\alpha(30) = 0\%$

Lifecycle strategy

Amount at retirement age (t=N) : first case : $\boxed{N > M}$

(*accumulation phase*)

$$S(N,N) = S(0).\exp\{N(m(N), V^2(N))\}$$

with :

$$m(N) = r.N + \alpha.(\delta - r).(N - M/2) - \sigma^2\alpha^2(N - 2M/3)/2$$
$$V^2(N) = \sigma^2\alpha^2(N - 2M/3)$$

Mean :

$$ES(N,N) = S(0).\exp(r\,N + \alpha.(\delta - r)(N - M/2))$$

Lifecycle strategy

Amount at retirement age (t=N) : second case : $\boxed{N < M}$

(*consolidation phase*)

$$S(N,N) = S(0).\exp\{N(m(N),V^2(N))\}$$

with :

$$m(N) = r.N + \alpha.(\delta - r).N^2/2M - \sigma^2\alpha^2 N^3/6M^2$$
$$V^2(N) = \sigma^2\alpha^2 N^3/3M^2$$

Mean :

$$ES(N,N) = S(0).\exp(r\,N + \alpha(\delta - r)N^2/2M)$$

3. Risk Measure Approach

Maturity guarantee

Comparison of these 3 strategies :

Matching between the final asset at maturity S(N,N) and the minimum maturity guarantee L(N)

$$\text{Asset}: \quad S(N,N)$$

$$\downarrow\uparrow$$

$$\text{Liability}: \quad L(N) = e^{r_G . N}$$

$$(r_G = \text{guaranteed rate at maturity})$$

Probability of default

In the 3 considered strategies, the asset at maturity can be written as :

$$S(N,N) = S(0).e^{N(m(N),V^2(N))}$$

Then the probability of default at maturity becomes :

$$\Psi(N) = P(S(N,N) < L(N))$$
$$= P(e^{N(m(N),V^2(N))} < e^{r_G N})$$
$$= \Phi\left(\frac{r_G N - m(N))}{V(N)}\right) = NORMSDIST\left(\frac{r_G N - m(N))}{V(N)}\right)$$

Probability of default - constant

Strategy 1 : constant proportion portfolio :

$$\Psi_1(N) = \Phi\left(\sqrt{N}.\frac{r_G - (r + \gamma.(\delta - r) - \gamma^2\sigma^2/2)}{\gamma\sigma}\right)$$

$$= \Phi(\sqrt{N}.a(\gamma))$$

The number a(γ) is in general negative .
So this probability decreases (deeply!) with the maturity N.

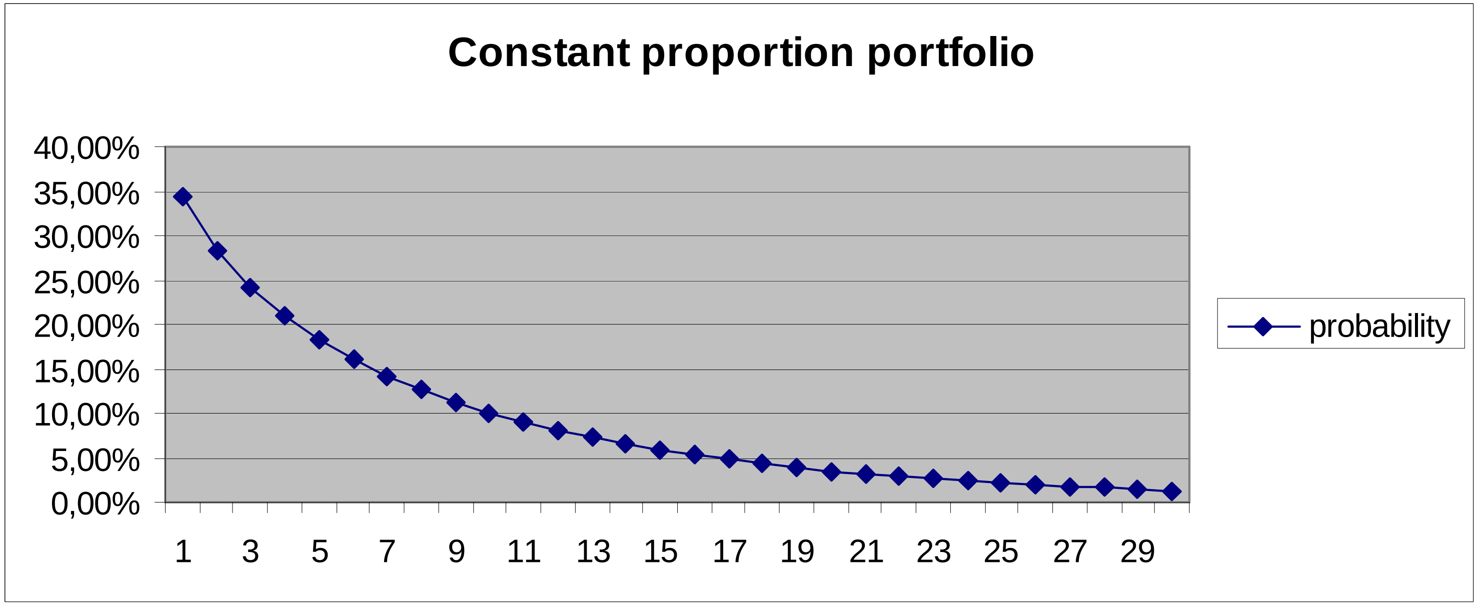
Probability of default - constant

EXAMPLE :

$$\alpha(t) = 70\%$$

$$\delta = 7\% \quad \sigma = 15\%$$

$$r = 3\% \quad r_G = 1\%$$



Probability of default - linear

Strategy 2 : linear decreasing strategy :

$$\Psi_2(N) = \Phi(\sqrt{3N}.\frac{r_G - (r + \beta.(\delta - r)/2 - \beta^2\sigma^2/6}{\beta\sigma})$$

$$= \Phi(\sqrt{N}.b(\beta))$$

The number b(β) is in general negative .
So this probability decreases also with the maturity N.

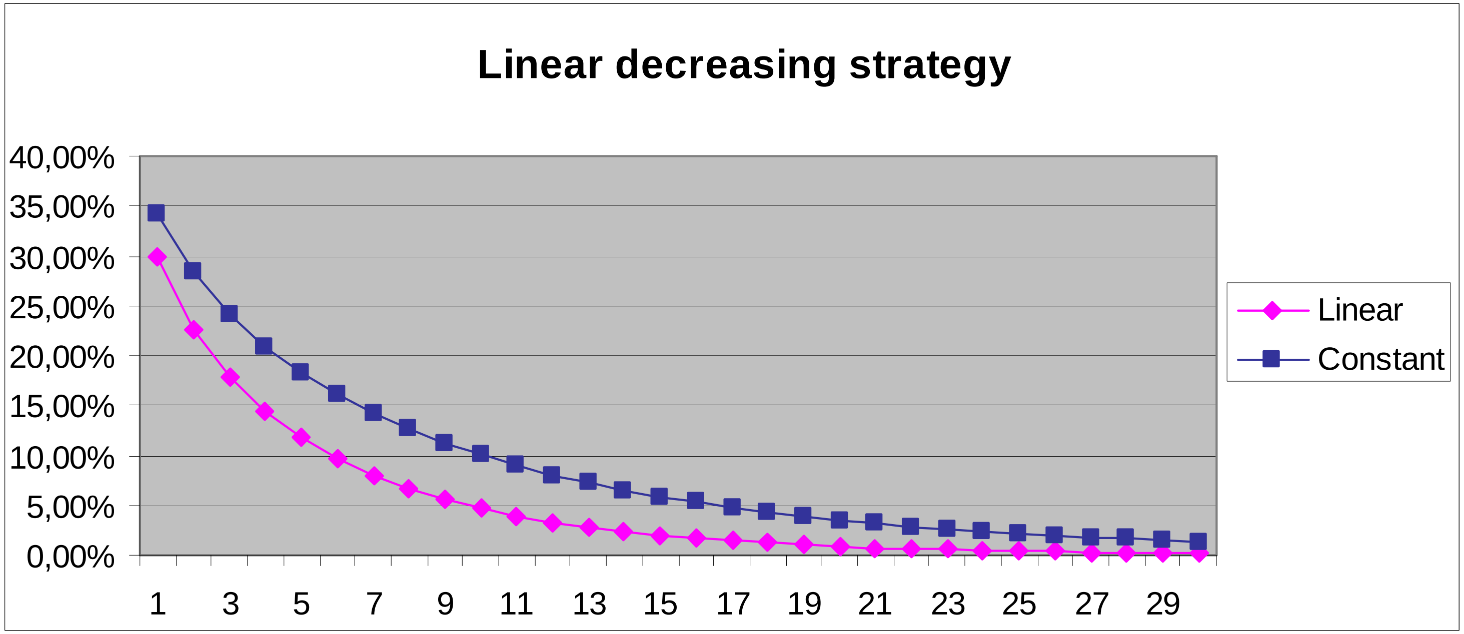
Probability of default - linear

EXAMPLE :

$$\alpha(t) = 70\%$$

$$\delta = 7\% \quad \sigma = 15\%$$

$$r = 3\% \quad r_G = 1\%$$



Probability of default - lifecycle

Strategy 3 : life cycle strategy :

N < M :

$$\Psi_3(N) = \Phi\left(M\sqrt{3}.\frac{r_G N - (rN + \alpha.(\delta - r)N^2/2M - \sigma^2\alpha^2N^3/6M^2)}{\alpha\sigma\sqrt{N^3}}\right)$$

N > M

$$\Psi_3(N) = \Phi\left(\frac{r_G N - (rN + \alpha(\delta - r).(N - M/2) - \sigma^2\alpha^2(N - 2M/3)/2)}{\alpha\sigma\sqrt{N - 2M/3}}\right)$$

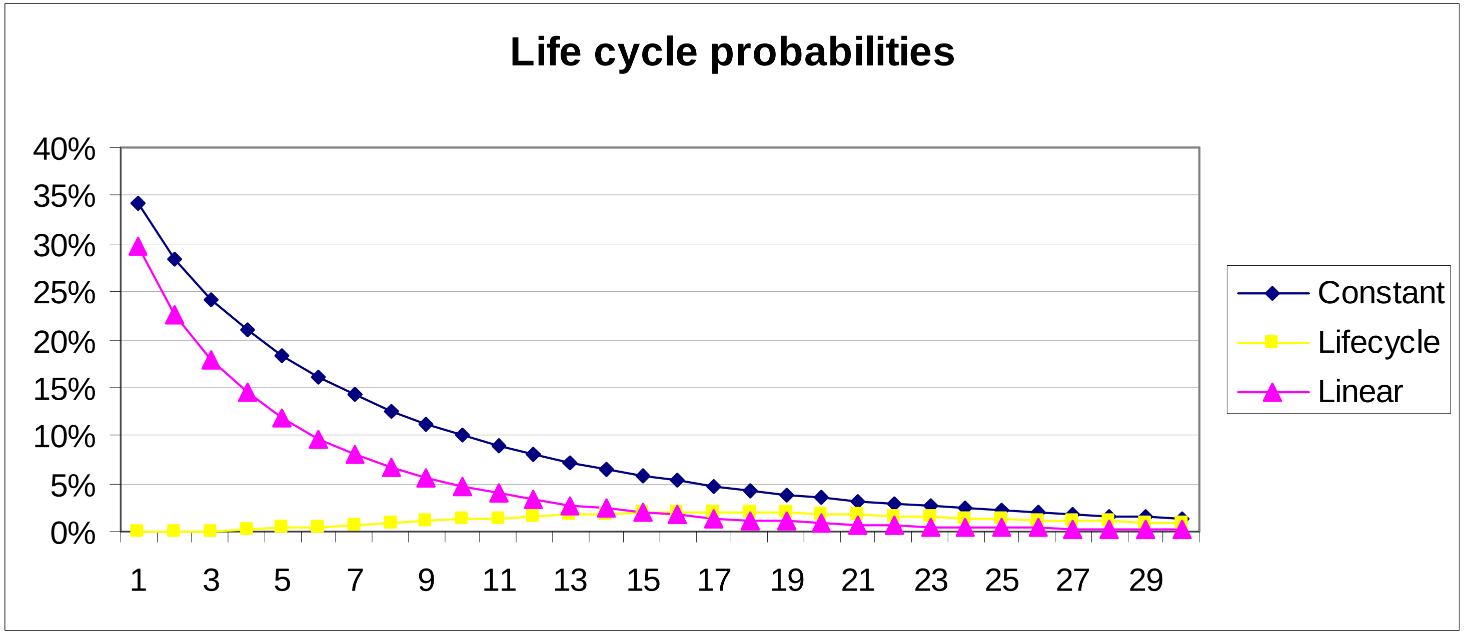
Probability of default – life cycle

EXAMPLE :

$$\alpha(t) = 70\%$$

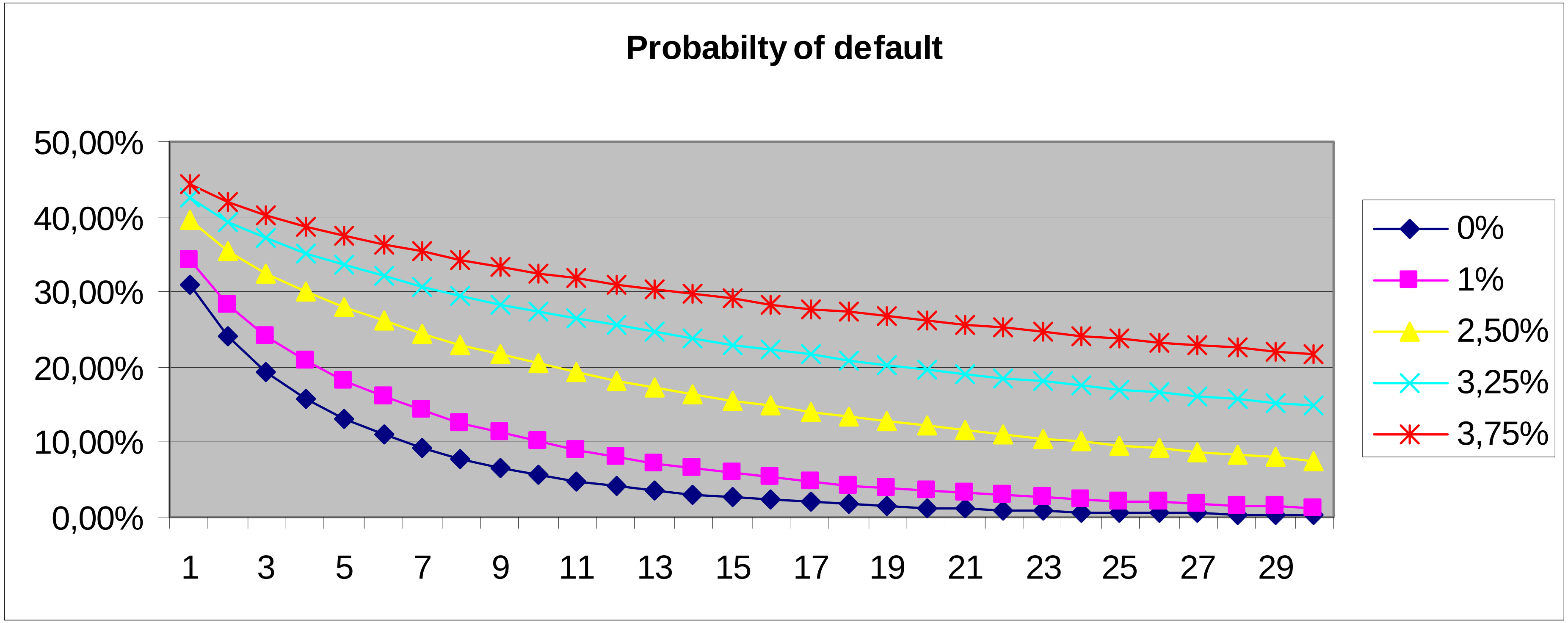
$$\delta = 7\% \quad \sigma = 15\%$$

$$r = 3\% \quad r_G = 1\% \quad M = 15$$



Constant proportion portfolio - probability

(for different levels of guarantee)



Value at risk - constant

We define the solvency level C(N) as the initial amount of equity invested in the riskless asset such that at maturity :

$$\boxed{P(S(N,N) + C(N).e^{rN} < L(N)) = \varepsilon_N}$$

where :

$$S(N,N) = e^{N(m(N),V^2(N))}$$

$$L(N) = e^{r_G N}$$

$$\varepsilon_N = \text{safety level (for instance : 0,5\%)}$$

Value at risk-constant

The solvency capital is then given explicitly by :

$$C(N) = e^{(r_G - r)N} - e^{-rN}.e^{m(N)+z_{\varepsilon_N}.V(N)}$$

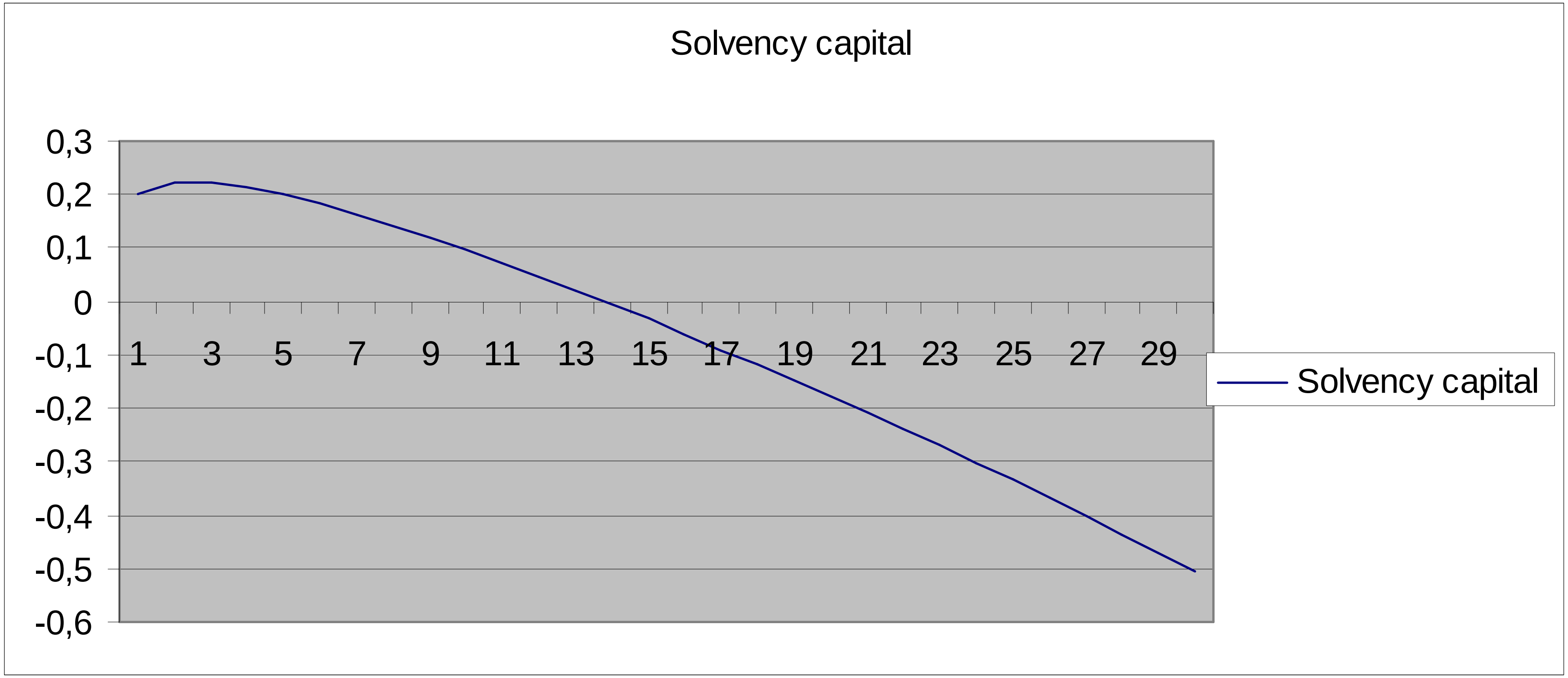
(where : $P(N(0,1) < z_\varepsilon) = \varepsilon$)

For instance for the constant proportion portfolio :

$$C(N) = e^{(r_G - r)N} - e^{-rN}.e^{rN+\gamma(\delta-r)N-\gamma^2\sigma^2 N/2+z_{\varepsilon_N}.\gamma\sigma\sqrt{N}}$$

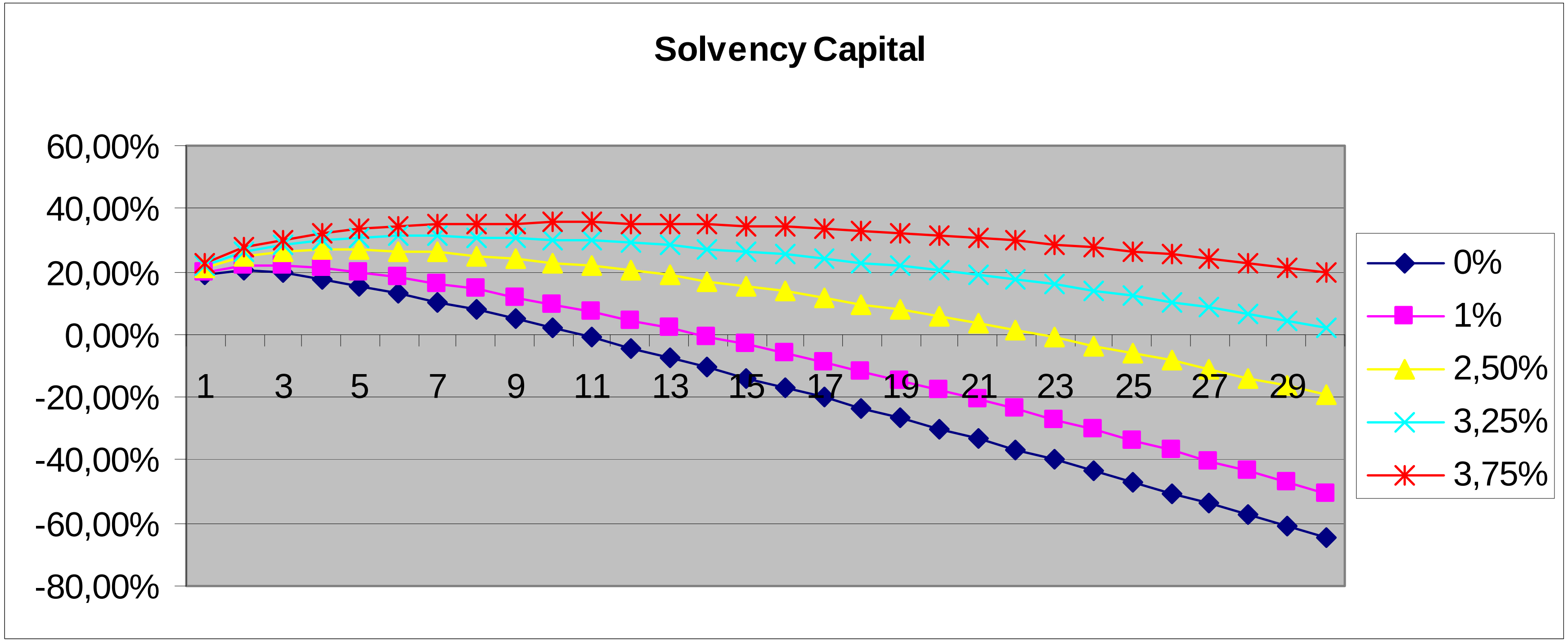
$$1-\varepsilon_N = (0.995)^N$$

$$r_G = 1\% \qquad r = 3\%$$
$$\delta = 7\% \qquad \gamma = 70\%$$
$$\sigma = 15\%$$



Constant proportion portfolio - solvency

(for different levels of guarantee)



4. Probability of Ruin Approach

4.1. Probability of ruin without capital

We try to compute the probability of non ruin without any extra capital :

$$P(N) = P(A(t) \geq L(t),\ \forall t \in [0, N])$$

With a pure equity strategy first :

$$A(t) = \exp(\delta t + \sigma w(t) - \sigma^2 t / 2)$$
$$L(t) = e^{r_G t}$$

This probability can be written using a minimum condition on a geometric Brownian motion :

$$P(A(t) - L(t) \geq 0,\ \forall\, t \in [0, N])$$

$$= P(\ e^{(\delta - \sigma^2/2)t + \sigma w(t)} \geq e^{r_G t}, \forall t \in [0, N])$$

$$= P(\min_{0 \leq s \leq N} e^{(\delta - r_G - \sigma^2/2)s + \sigma w(s)} \geq 1)$$

$$= 1 - P(\min_{0 \leq s \leq N} e^{(\delta - r_G - \sigma^2/2)s + \sigma w(s)} < 1)$$

Distribution function of the minimum of a geometric Brownian motion

Theorem :
Law of the minimum of a geometric Brownian motion :

Let :

$$S(t) = e^{(\mu-\sigma^2/2)t+\sigma.w(t)}$$

$$z = \min_{0\le s\le t}\{S(s)\}$$

$$0 < L \le S(0)$$

Then :

$$\boxed{P(z \le L) = \Phi(d_1) + L^{\frac{2\mu}{\sigma^2}-1}.\Phi(d_2)}$$

$$\text{with}: d_{1,2} = \frac{\ln L \mp (\mu - \sigma^2/2)t}{\sigma\sqrt{t}}$$

Computation of the probability of non ruin :

$$P(N) = P(A(t) - L(t) \geq 0,\ \forall t \in [0, N])$$

$$= 1 - P(\min_{0 \leq s \leq N} e^{(\delta - r_G - \sigma^2/2)s + \sigma w(s)} < 1)$$

$$= 1 - (\Phi(\frac{-(\delta - r_G - \sigma^2/2)N}{\sigma\sqrt{N}} + \Phi(\frac{(\delta - r_G - \sigma^2/2)N}{\sigma\sqrt{N}})$$

$$= 0$$

Under these assumptions the ruin is certain !!!

Variant 1 : partial solvency requirement before maturity (surrender charges in life insurance)

$$P\left(A(t) \geq e^{r_G t} ; \forall t \in [0, N]\right)$$

Liquidity penalty

$$\boxed{P\left(A(t) \geq \rho(t).e^{r_G t} ; \forall t \in [0, N]\right)}$$

ρ increasing

$\rho(t) \leq 1$

$\rho(N) = 1$

Example:

$\rho(s) = e^{-\lambda(N-s)}$

$(\lambda > 0)$

The solvency requirement becomes then :

$$\boxed{P(N,\lambda) = P(A(t) \geq e^{-\lambda N}.e^{(r_G+\lambda)t} ; \forall t \in [0,N])}$$

Introducing the minimum :

$$P(N,\lambda)$$

$$= P(\min_{0 \leq s \leq N} e^{(\delta - r_G - \lambda - \sigma^2/2)s + \sigma w(s)} \geq e^{-\lambda N})$$

$$= 1 - P(\min_{0 \leq s \leq N} e^{(\delta - r_G - \lambda - \sigma^2/2)s + \sigma w(s)} < e^{-\lambda N})$$

L < 1

The probability of non-ruin becomes now :

$$P(N,\lambda) = P(A(t) \geq e^{-\lambda(N-t)}.L(t),\ \forall t \in [0,N])$$

$$= 1 - \{\Phi(\frac{-\lambda N - (\delta - r_G - \lambda - \sigma^2/2)N}{\sigma\sqrt{N}})$$

$$+ \left(e^{-\lambda N}\right)^{\frac{2(\delta - r_G - \lambda)}{\sigma^2} - 1}.\Phi(\frac{-\lambda N + (\delta - r_G - \lambda - \sigma^2/2)N}{\sigma\sqrt{N}})\}$$

The probability of ruin can be written:

$$\Psi(N,\lambda) = 1 - P(N,\lambda)$$

$$= \Phi\left(\frac{-(\delta - r_G - \sigma^2/2)N}{\sigma\sqrt{N}}\right)$$

$$+ \left(e^{-\lambda N}\right)^{\frac{2(\delta - r_G - \lambda)}{\sigma^2} - 1} . \Phi\left(\frac{(\delta - r_G - 2\lambda - \sigma^2/2)N}{\sigma\sqrt{N}}\right)$$

$$= \underbrace{\Phi(a\sqrt{N})}_{\textit{Probability of default at maturity}} + \underbrace{(e^{-\lambda N})^{\frac{2(\delta - r_G - \lambda)}{\sigma^2} - 1} \Phi((-a - 2\lambda/\sigma)\sqrt{N})}_{\textit{Price of non ruin before maturity}}$$

Constant portfolio allocation :

We can also analyze the influence of the asset allocation on this last probability of non ruin :

$$\delta \to \gamma\delta + (1-\gamma)r$$

$$\sigma \to \gamma\sigma$$

$$P(N,\lambda,\gamma) = P(A(t) \geq e^{-\lambda(N-t)}.L(t),\ \forall t \in [0,N])$$

$$= 1 - \{\Phi(\frac{-((\gamma\delta + (1-\gamma)r\ - r_g) - \gamma^2\sigma^2/2)N}{\gamma\sigma\sqrt{N}}$$

$$+ \left(e^{-\lambda N}\right)^{\frac{2((\gamma\delta+(1-\gamma)r\ - r_G - \lambda)}{\gamma^2\sigma^2} - 1} .\Phi(\frac{((\gamma\delta + (1-\gamma)r - r_G) - 2\lambda - \gamma^2\sigma^2/2)N}{\gamma\sigma\sqrt{N}})\}$$

Variant 2 : Fair Valuation approach

Another possibility for the liability function is to introduce the fair value of the final liability (*present value at the risk free rate*):

$$L(N) = e^{r_G N}$$

$$L(t) = e^{-r\,(N-t)}.L(N) = e^{-r\,(N-t)}.e^{r_G N}$$

$$= e^{-(r - r_G)(N-t)}.e^{r_G t}$$

$$= e^{-\lambda^*(N-t)}.e^{r_G t}$$

This is a particular case of variant 1 with exponential penalty parameter given by :

$$\boxed{\lambda^* = r - r_G}$$

(spread of rates)

This coefficient is positive if the risk free rate is strictly higher than the guaranteed rate.

Then the probability of ruin becomes for a constant strategy:

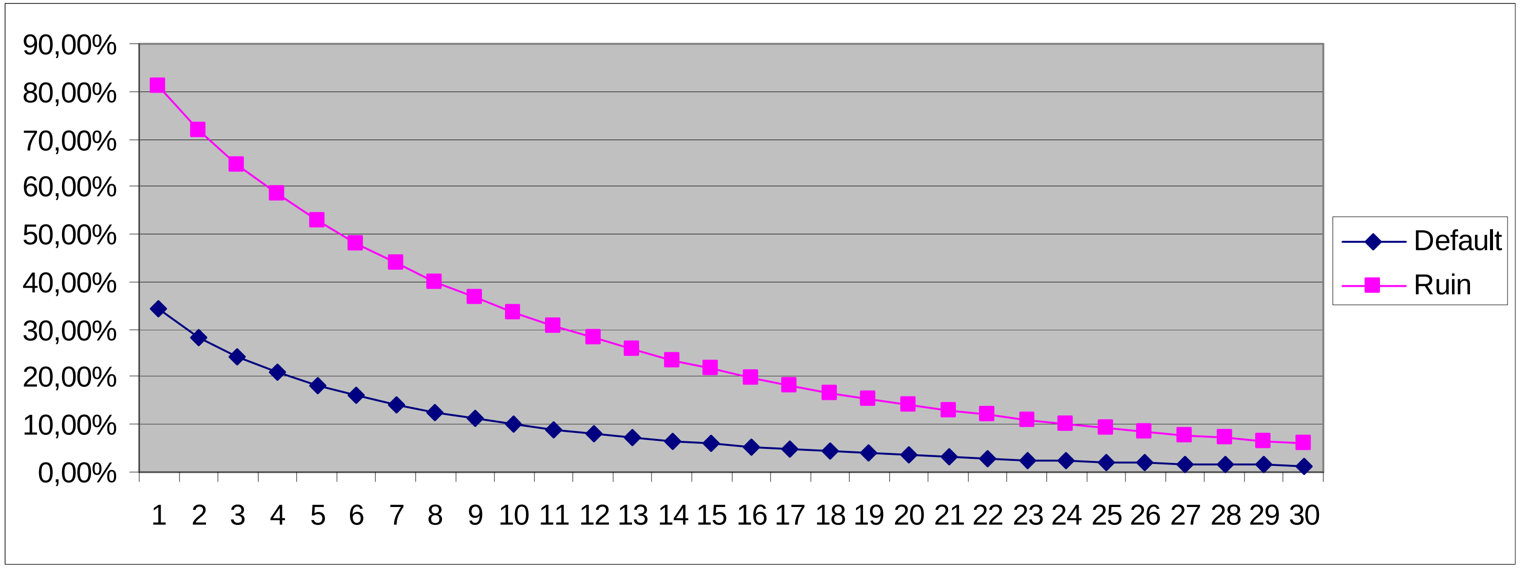
$$\Psi(N,\gamma) = \Phi(a(\gamma)\sqrt{N}) + (e^{-(r-r_G)N})^{\frac{2\gamma(\delta-r)}{\gamma^2\sigma^2}-1}\Phi((-a(\gamma)-2(r-r_G)/\gamma\sigma)\sqrt{N})$$

If the two rates are close , the coefficient λ is nearly zero and this probability is nearly equal to one (for every time horizon).

Probability of ruin – constant *(fair valuation approach)*

EXAMPLE :

$$\alpha(t) = 70\% \quad \lambda = 2\%$$
$$\delta = 7\% \quad \sigma = 15\%$$
$$r = 3\% \quad r_G = 1\%$$



4.2. Probability of ruin with riskless capital

We consider now the probability of non ruin, starting with a positive solvency capital assumed to be invested in a riskless asset and the contribution invested in the risky asset :

$$P(N,C) = P(A(t) + C.e^{rt} \geq L(t),\ \forall t \in [0,N])$$

Or with a partial solvency requirement :

$$P(N,\lambda,C) = P(\underbrace{A(t) + C.e^{rt}}_{\text{effect capital}} \geq \underbrace{e^{-\lambda(N-t)}L(t)}_{\text{effect product}},\ \forall t \in [0,N])$$

Computation of the probability of non ruin :

In general the problem is not so trivial and can no more be written as the law of the minimum of a geometric Brownian motion …

$$P(N,\lambda,C) = P(A(t) + C.e^{rt} \geq e^{-\lambda(N-t)}L(t),\ \forall\, t \in [0,N])$$

$$= P(e^{(\delta-\sigma^2/2)t+\sigma w(t)} \geq e^{-\lambda N}e^{(r_G+\lambda)t} - C.e^{rt},\ \forall t \in [0,N])$$

We can obtain an explicit form in the fair value case :

$$\boxed{\lambda^* = r - r_G}$$

Then we can write :

$$P(N,\lambda^*,C) = P(e^{(\delta-r_g-\lambda^*-\sigma^2/2)t+\sigma w(t)} \geq \underbrace{e^{-\lambda^* N} - C}_{L},\ \forall t \in [0,N])$$

Using the previous results we obtain for the probability of ruin :

$$\Psi(N,\lambda^*,C) = 1 - P(N,\lambda^*,C) = 1 - P(A(t) \geq e^{-\lambda^*(N-t)}.L(t),\ \forall t \in [0,N])$$

$$= \Phi\left(\frac{\ln(e^{-(r-r_G)N} - C) - (\delta - r - \sigma^2/2)N}{\sigma\sqrt{N}}\right.$$

$$\left. + \left(e^{-(r-r_G)N} - C\right)^{\frac{2(\delta-r)}{\sigma^2}-1}.\Phi\left(\frac{\ln(e^{-(r-r_G)N} - C) + (\delta - r - \sigma^2/2)N}{\sigma\sqrt{N}}\right)\right\}$$

With : $\Psi(N,0,0) = 1$

Without capital and without spread you are sure to be ruined !!!

5. Introduction of the Longevity Risk

Introduction of a longevity condition :

The pension liability is paid at retirement age only in case of survival .

The guarantee takes now into account a pre defined life table based on a deterministic mortality intensity :

$$
\begin{aligned}
{}_N p_x &= \Pr \text{oba to survive } N \text{ years between } x \text{ and } x+N \\
&= \exp - \int_0^N \mu_{x+s}\, ds
\end{aligned}
$$

The minimum pension benefit to pay in case of survival at retirement age is now given by :

$$B(N) = \frac{e^{r_G N}}{{}_N p_x} = e^{\int_0^N (r_G + \mu_{x+s})\, ds}$$

For this “a priori” life table we will use a classical GOMPERTZ law :

$$\mu_{x+s} = \mu_x . e^{\beta s}$$

On the other hand we assume that the real death intensity will follow an Ornstein- Uhlenbeck stochastic process given by:

$$d\lambda_x(t) = a.\lambda_x(t)\,dt + \rho\,dw_x(t)$$

The solution of this classical SDE is given by :

$$\lambda_x(t) = \lambda_x e^{at} + \rho \int_0^t e^{a(t-s)}\,dw_x(s)$$

Assuming as before an initial contribution of 1 € paid at time t=0 , the asset – liability structure for a pure equity strategy (or a constant proportion portfolio), is now given by :

$$A(N) = e^{(\delta - \sigma^2/2)N + \sigma . w(N)}$$

$$L(N) = B(N).e^{-\int_0^N \lambda_x(s)ds} = e^{r_G N}.e^{\int_0^N (\mu_{x+s} - \lambda_x(s)).ds}$$

Asset and Liability are now both stochastic !!!

Probability of default at maturity

(taking into account market and longevity risks)

$$P(A(N) < L(N)) = P(e^{(\delta-\sigma^2/2)N+\sigma w(N)} < e^{r_G N + \int_0^N (\mu_{x+s} - \lambda_x(s)).ds})$$

$$= P(\sigma w(N) + \int_0^N \lambda_x(s)\,ds < (r_G - \delta + \sigma^2/2)N + \int_0^N \mu_{x+s} ds)$$

$$= P(\ \Theta(N) < f(N))$$

Where

$$\Theta(N) = \sigma\, w(N) + \int_0^N \lambda_x(t)dt$$

$$= \sigma\, w(N) + \int_0^N \{\lambda_x e^{at} + \rho \int_0^t e^{a(t-s)} dw_x(s)\}\, dt$$

$\Theta(N)$ is normally distributed :

$$\Theta(N) \propto N(\, m(N), \sigma^2(N))$$

The probability of joined default becomes :

$$P(A(N)) < L(N)) = \Phi\left(\frac{(r_G - \delta + \sigma^2 / 2)N + \int_0^N \mu_{x+s} ds - m(N)}{\sigma(N)}\right)$$

Value of the 2 moments :

1° mean :

$$m(N) = E\Theta(N) = \int_0^N \lambda_x e^{at}\, dt = \lambda_x \frac{e^{aN} - 1}{a}$$

$$\int_0^N \mu_{x+s} ds = \int_0^N \mu_x e^{\beta s} ds = \mu_x \frac{e^{\beta N} - 1}{\beta}$$

So the probability of default becomes :

$$P = \Phi\left(\frac{(r_G - \delta + \sigma^2/2)N + (\mu_x \frac{e^{\beta N} - 1}{N} - \lambda_x \frac{e^{aN} - 1}{N})}{\sigma(N)}\right)$$

Spread of rates

Spread of mortality

Value of the 2 moments :

2°) variance :

Assumption : independence between financial and longevity risks

$$\sigma^2(N) = \mathrm{var}\ \Theta(N)$$

$$= \sigma^2 N + \rho^2\ E\left(\left(\int_0^N \int_0^t e^{a(t-s)} dw_x(s)\right)^2\right)$$

$$= \sigma^2 N + \frac{\rho^2}{2a^3}(2aN + e^{2aN} - 4e^{aN} + 3)$$

6. Extension to DB schemes

Model for a DB Scheme

We consider a ***Final Salary DB Scheme***

At retirement age : lump sum expressed as a multiple of the last salary .

For instance : *at 65* : Liability side :

$$C = \frac{N}{40} \; k.S(65)$$

where: N = years of service at retirement age
S(65) = last salary before retirement

Following the IAS 19 norms , we use as funding method the ***Projected Unit Credit Cost*** .

In order to compute the contribution (" *Normal Cost* ") we need the following assumptions :

1°) *discount rate* (risk free rate) : r

2°) *life table* : we will assume here no mortality before retirement (no longevity risk considered here)

3°) *salary scale projection* : stochastic model :

$$S(t) = S(0).e^{\mu t + \eta z(t) - \eta^2 t/2}$$

$$z(t) = \text{standard brownian motion}$$
$$\mu = \text{average salary increase}$$
$$\eta = \text{volatility on salary}$$

The contribution for one year of service in a best estimate pricing is then (assuming S(0) =1) :

$$NC_0 = \frac{1}{40}\, k\, e^{(\mu - r)T}$$

where : T = residual time to retirement

Possible loading on contribution (Normal Cost) :

We can introduce an eventual loading on contribution:

$$NC_0 = \frac{1}{40} k\, e^{(\mu - r)T} . (1+\beta)$$

Loading on Contribution

Asset side :

The contribution is assumed to be invested in a geometric Brownian asset :

$$A(t) = A(0).e^{\delta t + \sigma w(t) - \sigma^2 t/2}$$

$$w(t) = \text{standard brownian motion}$$

$$\text{corr}(w(t), z(t)) = \rho t$$

$$\delta = \text{mean return of the risky asset}$$

$$\sigma = \text{volatility}$$

$$A(0) = NC_0$$

(correlation between inflation and financial returns !!!)

Asset Liability risk management :

We compare at retirement age for this specific year of service the corresponding Asset and Liability :

Liability at maturity :

Inflation risk

$$L(T) = \frac{1}{40} k\, e^{\mu T + \eta z(T) - \eta^2 T/2}$$

Asset at maturity :

Investment risk

$$A(T) = \frac{1}{40} k(1+\beta).e^{(\mu - r)T}.\, e^{\delta T + \sigma w(T) - \sigma^2 T/2}$$

Asset and Liability are again both stochastic !!!

Probability of default at maturity :

$$\Psi(T) = P(A(T) < L(T))$$

$$\Psi(T) = P((1+\beta)e^{(\mu-r)T}.e^{\delta T}.e^{\sigma w(T)-\sigma^2 T/2} < e^{\mu T}e^{\eta z(T)-\eta^2 T/2})$$
$$= P(Y(T) < M)$$

With :

$$M = (r - \delta + \sigma^2/2 - \eta^2/2)T - \ln(1+\beta)$$
$$Y(T) = \sigma\, w(T) - \eta\, z(T)$$
$$= N(0, \bar{\sigma}^2)$$
$$\bar{\sigma}^2 = (\sigma^2 + \eta^2 - 2\rho\sigma\eta)$$

Probability of default at maturity :

So finally :

$$\Psi(T) = \Phi(d(T))$$

With :

$$\Phi = \text{distribution function } N(0,1)$$

$$d(T) = \frac{1}{\sigma}\left((r - \delta + \sigma^2/2 - \eta^2/2)\sqrt{T} - \ln(1+\beta)/\sqrt{T}\right)$$

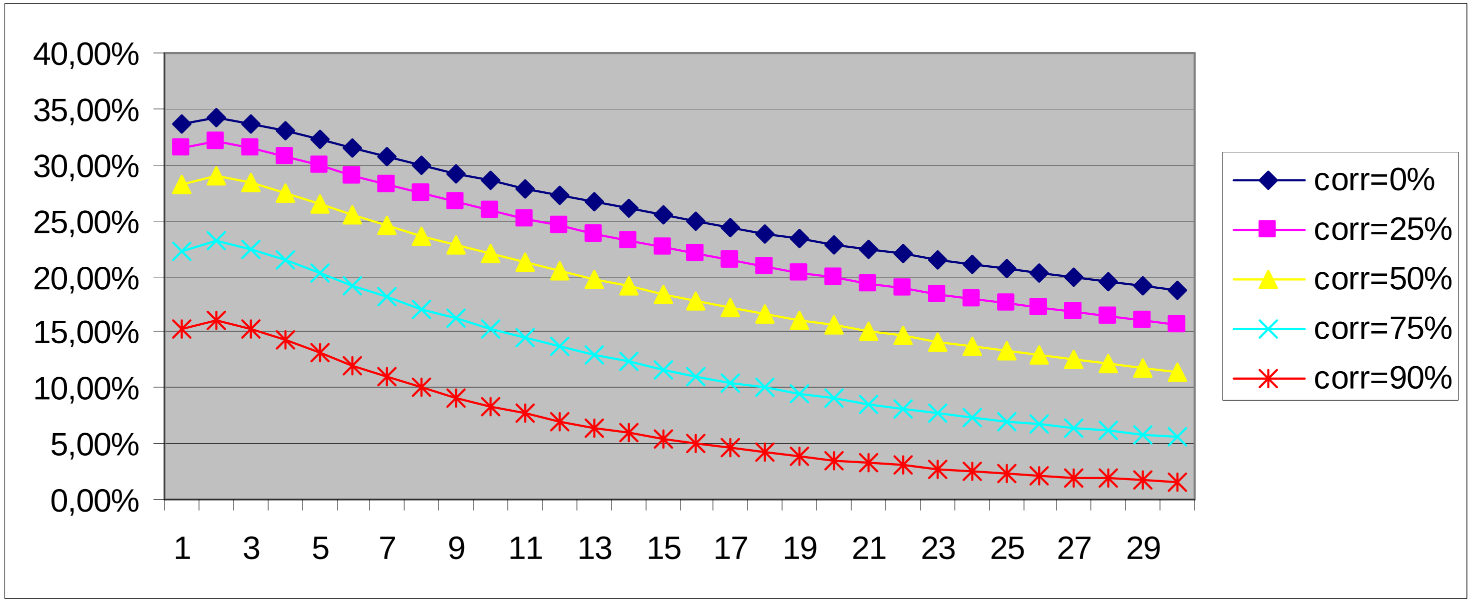
Rem.: independent of salary increase !

Probability of default : correlation impact

$$r = 3\% \qquad \delta = 7\%$$

$$\mu = 5\% \qquad \sigma = 15\%$$

$$\eta = 10\% \qquad \beta = 5\%$$



Next research

- Introduction of jumps
- Mean reverting inflation
- Generalization to LEVY processes
- Guarantees on periodical contributions
- Adaptation to PAYG schemes (*NDC technique*) with a demographic stochastic entrance process

References

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Thank you !!!

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