Stochastic calculus applied to the estimation of loss reserves in mortgage insurance

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- What is and what is not mortgage insurance in Mexico?
- 3 times model (Black and Scholes)
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Objective of the paper

- To explain some technical aspects of Mortgage Insurance in a formal document.
- To apply some techniques of risk management to the actuarial science.
- To propose a methodology for estimation of losses based on information of the risks.
- To solve some operating problems of insurance companies related to information quality.
- To avoid stochastic simulation! Have you tried to get a closed formula before simulation?

Binomial Model

- Let M be a stochastic process which represents the number of payments that has not been paid by a creditor of a mortgage loan. This process will be called:

DELINQUENCY INDEX

$$M = (M_t)_{t \in T} \qquad T = \{0,1,2,3,4 \ldots N-1\}$$

Binomial Model

- Other ingredients:

  "Insurance function" which represent the payment of an insurance company depending on the delinquency index:

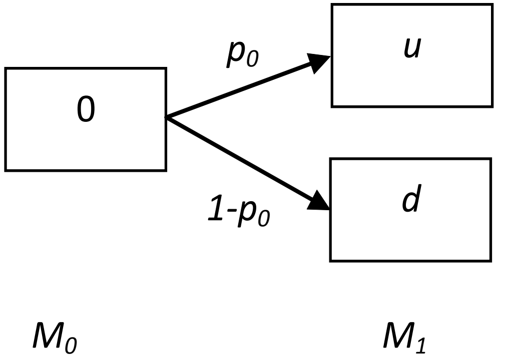
$$f(M_t): R \to R^+$$

"Risk free cash bond": Stochastic process.

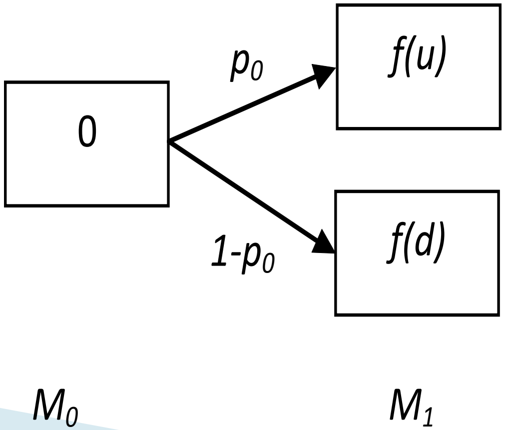
$$B_t = B_0 e^{rt} \ with \ B_o = 1$$

Binomial Model

- Finally the trees! (sorry, just a branch)
  For the delincuency index:



- For the insurance function:



Binomial Model

- With these trees it is possible to find and present value of the expected value of the losses of a company:

$$E = E[f(M_t)] = e^{-r}[p_0 \times f(u) + (1 - p_0) \times f(d)]$$

- We can interpret this formula considering the Kolmogorov´s strong law of large numbers. According to that law, if the company has a portfolio of insurance policies sufficiently large, that company can expect a loss given by this formula.

Let´s complicate de model

- Now consider and Asset which value “depends” on mortgage loans performance, that is, a Morgage Back Security: MBS “Trendy word”.

$$A(M_t) \qquad A: R \to R$$

- Supose that the Mortgage insurance company “wishes” to hedge or to match its losses by using a portfolio containing Mortgage back securities and the risk free cash bond. That is:

$$C_t = aA(M_t) + bB_t$$

Free arbitrage valuation and change of probability measure

- In order to carry out that “hedging estrategy”, it is necessary to construct the following system of equations:

$$aA(u) + bB_0e^r = f(u) \text{ and } aA(d) + bB_0e^r = f(d)$$

- If we solve it and substitute in the portfolio expresion, we have:

$$C_0 = e^{-r}\left[\left(\frac{A(0)e^r - A(d)}{A(u) - A(d)}\right)f(u) + \left(\frac{A(u) - A(0)e^r}{A(u) - A(d)}\right)f(d)\right]$$

Free arbitrage valuation and change of probability measure

- Or..

$$C_0 = e^{-r}[q_0 f(u) + (1 - q_0) f(d)]$$

- With

$$q_0 = \frac{A(0)e^{r} - A(d)}{A(u) - A(d)}$$

- In this way we have found another expected value of the losses of the company by making a "change of probability" let´s say Q. This process is used in financial derivatives valuation.

Important result.

- The existence of the probability measure $\mathbb{Q}$ is equivalent to state that the present value of the asset that depends on the "delinquency index" MBS process, is a "MARTINGALE". In our example:

$$\mathrm{E}_{\mathbb{Q}}[e^{-r}A(M_t)|\mathcal{F}_0] = e^0 A(0)$$

- Remember that an adaptive process is a Martingale if:

$$\mathrm{E}_{\mathbb{III}}\left[M_j|\mathcal{F}_i\right] = M_i \quad si \quad j \geq i$$

Some concepts

- To make a generalization of the model it is important to be familiar with:
  - Probability measure, Filtration (very important!!), Financial Claim, Conditional expectation, Adaptive and previsible processes, Martingale.

- This concepts can be used to develop the binomial model for n branches in the same way as before. Nevertheless the important result is that by calculating the expected value of the losses of a mortgage insurancre company and making the change of probability, the present value of the asset that depends on the "delinquency index" process, is a "MARTINGALE".

What is and what is not mortgage insurance in Mexico?

- It is:
  - Tool for transferring credit risk in mortgages.
  - will pay the benefit only when financial institution takes over the house due to default.
  - It covers just a portion of the OB + interest up to taking over the house.
- It is not:
  - Unemployment insurance.
  - Financial warranty insurance (MBS)
  - Classical P&C insurance.
- It Started in november 30 2006 when the regulator enacted the "Rules for Mortgages Insurances"

What is and what is not mortgage insurance in Mexico?

- A mortgage insurance company has to constitue:
  - Mathematical Reserve (actuarial or premium reserve)
  - Catastrophic Reserve (50% of the risk premium + i)
  - Capital (depending on seasoning and LTV)
  - "Claims reserves" (Outstanding reserves). Based on the "DELINQUENCY INDEX". That is this reserve can be calculated as a expected vaule of the losses of a company.

- The rest of the presentation will be related with claims reserve.

Imagine...

- Please, imagine that the two branches example tree has now an infinite number of branches.
- That idea leads us to important concepts
  - Brownian motion
  - Difussions
  - Stochastic calculus: “Black and Scholes Model”
  - Ito lemma
  - Ito processes (martingale if drift is 0):

$$M_t = M_0 + \int_0^t \sigma_s dW_s + \int_o^t \mu_s ds$$

$$dM_t = \sigma_t dW_t + \mu_t dt$$

3 times model

- Supose that M "Delinquency index" is a geometric brownian motion:

$$\frac{dM_t}{M_t} = \sigma dW_t + \mu dt \quad \Leftrightarrow \quad M_t = M_{t_0} e^{\sigma(W_t - W_{t_0}) + \left(\mu - \frac{1}{2}\sigma^2\right)(t - t_0)}$$

- The risk free chash bond is another difussion:

$$\frac{dB_t}{B_t} = r dt \quad \Leftrightarrow \quad B_t = B_{t_0} e^{r(t-t_0)}$$

- The present value of the MBS which is a deterministic linear function:

$$D_t = B_t^{-1} A(M_t) = B_t^{-1} A_t M_t$$

3 times model

- This model is based in the fact that the present value of the MBS is a Martingale under a change of probabilty. So by using the following techniques of estochastic calculus:
  - Ito´s lemma
  - Martin Cameron Girsanov Theorem
- We get:

$$\frac{dD_t}{D_t} = \sigma d\widetilde{W}_t$$

- And:

$$\frac{dM_t}{M_t} = \sigma d\widetilde{W}_t + r dt$$

3 times model

- Now we calculate the expected losses as:

$$OPC_t = E_{\mathbb{Q}}\left[B_{T-t}^{-1} f(T) | \mathcal{F}_{t_0}\right] = E_{\mathbb{Q}}\left[B_{T-t}^{-1} \times \%Cob \times SI_T \times I_{M_t \geq 6} | \mathcal{F}_{t_0}\right]$$

- By applying the Black and Scholes Methodoly, we have:

$$OPC_t = \%Cob \times E_{\mathbb{Q}}[B_{T-t}^{-1} \times SI_T] \times \mathbb{Q}\left[\eta \leq d_2 | M_{t_0}\right] = \%Cob \times E_{\mathbb{Q}}[B_u^{-1} \times SI_{t+u}] \times \Phi(d_2)$$

- With:

$$T = t + u$$

$$d_2 = \frac{\ln \frac{M_{t_0}}{6} + \left(r - \frac{1}{2}\sigma^2\right)(t - t_0)}{\sigma\sqrt{t - t_0}}$$

3 times model

- 3 times in the model: information, valuation, time when the claim is paid. If u is a constant:

$$OPC_t = \%Cob \times e^{-ru} \times SI_{t+u} \times \Phi(d_2)$$

- By using other techniques of stochastic calculus (Ito´s lemma) it is possible to obtain the hedging portfolio:

$$P_t = \phi_t A(M_t) + \psi_t B_t = \phi_t A_t M_t + \psi_t B_t$$

$$\phi_t A_t = \frac{d}{dM_t}(\%Cob \times e^{-ru} \times SI_{t+u} \times \Phi(d_2)) = \%Cob \times e^{-ru} \times SI_{t+u} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}}$$

$$F_t = B_t^{-1} P_t = \phi_t D_t + \psi_t$$

Correlated brownian motion

- It is possible to adjust the "Delinquency index" defined in the 3 times model by using another brownian motion. If we define the following expresion as the "Total delinquency index" (both factors are geometric brownnian motion: $MT_t = M_t^1 M_t^2$ $dW_t^i dW_t^j = \begin{cases} dt & si\ i = j \\ \rho dt & si\ i \neq j \end{cases}$

- By using the integration by parts lemma it can be demonstrated that:

$$\frac{dMT_t}{MT_t} = \sigma_1 dW_t^1 + \sigma_2 dW_t^2 + (\mu_1 + \mu_2 + \sigma_1 \sigma_2 \rho) dt$$

Correlated brownian motion

- If we consider the "Total volatility" and the following process:

$$\Gamma = \sqrt{\sigma_1^2 + \sigma_2^2} \qquad N_t = \int_0^t \frac{\sigma_1}{\Gamma} dW_s^1 + \int_0^t \frac{\sigma_2}{\Gamma} dW_s^2$$

- We have another brownian model for the Total delinquency index:

$$\frac{dMT_t}{MT_t} = \Gamma dN_t + (\mu_1 + \mu_2 + \sigma_1 \sigma_2 \rho) dt$$

Correlated brownian motion

- And we can use the techniques used inthe 3 ties model:

$$OPC_t = \%Cob \times E_{\mathbb{Q}}[B_u^{-1} \times SI_{t+u}] \times \Phi(d_2)$$

$$d_2 = \frac{\ln \frac{MT_{t_0}}{6} + \left(r - \frac{1}{2}\Gamma^2\right)(t - t_0)}{\Gamma\sqrt{t - t_0}}$$

- We do not have to know the drift of the processes nor even the exact correlations of the brownian motion.

- It can be shown that: $\frac{dd_2}{d\sigma_2} < 0$

Other concepts

- This techniques can be used for:
  - Pricing (simulation is required)

$$A_x = \sum_{k=0}^{n-1} v^{k+1} (SA_{k+u}) p_x(k) q_{x+k} \qquad q_x = \mathbb{Q}[M_x \geq 6 | M_{x-1} < 6] = \Phi(d_2)$$

  - Mathematical Reserves

$$V_t = A_x(t) - \ddot{a}_x(n-t) P_x$$

  - Capital:

$$C_t = \Phi^{-1}(x) \times \sqrt{n \times Var_t} + n \times OPC_t$$

Results

- Applaying these concepts to sample of a real portfolio to estimate future losses we have:

Regulator	3 times model	CORRELATED BROWNIAN MOTION Var of M_t^2				
		Var 5%	Var 25%	Var 50%	Var 75%	Var 100%
1,167.12	785.45	740.39	623.52	533.10	467.29	415.23

- Also for the pricing:

Coverage	20%	25%	30%
Benefit premiums	1.04%	1.30%	1.56%
Level premium	0.00%	0.01%	0.01%

Conclusions

- Some techniques of stochastic calculus can be applied in the estimation of losses for mortgage insurances
- Special attention should be paid to the normality assumptions that the Black and Scholes model implies
- In the numerical results we can show that it is possible to have shorter reserves requirements.
- The 3 times model allows to remove the markovian approach of the current regulatory requirements in Mexico, it can be used to solve some operational problems
- Score models.