Correlation, tail dependence and diversification

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Agenda

1. Introduction

"Although it is an old idea, the measurement and allocation of diversification in portfolios of asset and / or liability risks is a difficult problem, which has so far found many answers. The diversification effect of a portfolio of risks is the difference between the sum of the risk measures of stand-alone risks in the portfolio and the risk measure of all risks in the portfolio taken together, which is typically non-negative, at least for positive dependent risks."

[Hürlimann (2009a), p. 325]

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1. Introduction

"Diversification arises when different activities complement each other, in the field of both return and risk. […] The diversification effect is calculated by using correlation factors. Correlations are statistical measures assessing the extend to which events could occur simultaneously. […] A correlation factor of 1 implies that certain events will always occur simultaneously. Hence, there is no diversification effect and two risks identically add up. Risk managers tend to say that such risks are perfectly correlated (i.e., they have a high correlation factor), meaning that these two risks do not actually diversify at all. A correlation factor of 0 implies that diversification effects are present and a certain diversification benefit holds."

[Doff (2007), p. 167f.]

1. Introduction

"By diversifiable we mean that if a risk category can be subdivided into risk classes and the risk charge of the total risk is not higher than the sum of the risk charges of each subrisk, then we have the effect of diversification. […] This effect can be measured as the difference between the sum of several capital charges and the total capital charge when dependency between them is taken into account."

[Sandström (2006), p. 188]

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1. Introduction

" 'Diversification effects' means the reduction in the risk exposure of insurance and reinsurance undertakings and groups related to the diversification of their business, resulting from the fact that the adverse outcome from one risk can be offset by a more favourable outcome from another risk, where those risks are not fully correlated. The Basic Solvency Capital Requirement shall comprise individual risk modules, which are aggregated […] The correlation coefficients for the aggregation of the risk modules […], shall result in an overall Solvency Capital Requirement […] Where appropriate, diversification effects shall be taken into account in the design of each risk module."

[Directive 2009/138/EC, (64) p. 7; (37) p. 24; Article 104, p. 52]

2. A short review of risk measures

Let $\mathfrak{X}$ be a suitable set of non-negative random variables X on a probability space $(\Omega, \mathcal{A}, P)$. A risk measure R on $\mathfrak{X}$ is a mapping $\mathfrak{X} \to \mathbb{R}^+$ with the following properties:

$P^X = P^Y \Rightarrow R(X) = R(Y)$	for all $X, Y \in \mathfrak{X}$	[distribution invariance]
$R(cX) = c\,R(X)$	for all $X \in \mathfrak{X}$ and $c \geq 0$	[scale invariance]
$R(X + c) = R(X) + c$	for all $X \in \mathfrak{X}$ and $c \geq 0$	[translation invariance]
$R(X) \leq R(Y)$	for all $X, Y \in \mathfrak{X}$ with $X \leq Y$	[monotonicity]

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2. A short review of risk measures

The risk measure is called *coherent*, if it additionally has the property:

$R(X+Y)\le R(X)+R(Y)$ for all $X,Y\in\mathfrak{X}$ [subadditivity]

This last property is the crucial point: it induces a diversification effect for *arbitrary* non-negative risks $X_1,\cdots,X_n$ (dependent or not) since it follows by induction that coherent risk measures have the property

$$R\left(\sum_{k=1}^{n}X_k\right)\le\sum_{k=1}^{n}R(X_k)$$ for any $n\in\mathbb{N}$.

2. A short review of risk measures

In what follows we shall use the term "(risk) *concentration* effect" as opposite to "*diversification* effect", characterized by the converse inequality

$R(X+Y) > R(X)+R(Y)$ for some $X, Y \in \mathfrak{X}$.

The popular standard deviation principle *SDP* which is sometimes used for tariffing in insurance is defined as

$SDP(X) = E(X) + \gamma\sqrt{Var(X)}$ for a fixed $\gamma > 0$ and $X \in \mathfrak{X} = \mathfrak{L}_+^2(\Omega, \mathcal{A}, P)$,

the set of non-negative square-integrable random variables on $(\Omega, \mathcal{A}, P)$.

SDP is *not* a risk measure because it is *not monotone*.

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2. A short review of risk measures

The risk measure used in Basel II/III and Solvency II is the Value-at-Risk VaR, being defined as a (typically high) quantile of the risk distribution:

$\text{VaR}_\alpha(X) := Q_X(1-\alpha)$ for $X \in \mathfrak{X}$ and $0 < \alpha < 1$,

where Q_X denotes the quantile function

$Q_X(u) := \inf\{x \in \mathbb{R} \mid P(X \le x) \ge u\}$ for $0 < u < 1$.

VaR is a proper risk measure, but not coherent in general.

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2. A short review of risk measures

The “smallest” coherent risk measure above VaR is the expected shortfall ES, which is in general defined as

$$\mathrm{ES}_\alpha(X) := \frac{1}{\alpha}\left\{E\left(X\cdot \mathbb{1}_{\{X\geq \mathrm{VaR}_\alpha(X)\}}\right) + \mathrm{VaR}_\alpha(X)\left[\alpha - P\left(X \geq \mathrm{VaR}_\alpha(X)\right)\right]\right\} \quad \text{for } 0<\alpha<1,$$

where $\mathbb{1}_A$ denotes the indicator random variable of some event (measurable set) A. In case that $P\left(X \geq \mathrm{VaR}_\alpha(X)\right)=\alpha$, this formula simplifies to

$$\mathrm{ES}_\alpha(X) = E\left(X \mid X \geq \mathrm{VaR}_\alpha(X)\right) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u(X)\,du.$$

3. A short review of copulas

A copula (in n dimensions) is a function C defined on the unit cube $[0,1]^n$ with the following properties:

- the range of C is the unit interval $[0,1]$;
- $C(\mathbf{u})$ is zero for all $\mathbf{u} = (u_1, \cdots, u_n)$ in $[0,1]^n$ for which at least one coordinate is zero;
- $C(\mathbf{u}) = u_k$ if all coordinates of $\mathbf{u}$ are 1 except the k-th one;
- C is n-increasing in the sense that for every $\mathbf{a} \leq \mathbf{b}$ in $[0,1]^n$ the volume assigned by C to the subinterval $[\mathbf{a},\mathbf{b}] = [a_1, b_1] \times \cdots \times [a_n, b_n]$ is nonnegative.

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3. A short review of copulas

A copula can alternatively be characterized as a multivariate distribution function with univariate marginal distribution functions that belong to a continuous uniform distribution over the unit interval $[0,1]$.

Every copula is bounded by the so-called Fréchet-Hoeffding bounds, i.e.

$$C_*(\mathbf{u}) := \max(u_1 + \cdots + u_n - n + 1, 0) \le C(u_1, \cdots, u_n) \le C^*(\mathbf{u}) := \min(u_1, \cdots, u_n).$$

The upper Fréchet-Hoeffding bound C^* is a copula itself for any dimension; the lower Fréchet-Hoeffding bound C_* is a copula in two dimensions only.

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3. A short review of copulas

If X is any real random variable, then

the random vector $\mathbf{X} = (X, X, \cdots, X)$ with n components possesses the upper Fréchet-Hoeffding bound C^* as copula

the random vector $\mathbf{X} = (X, -X)$ with two components possesses the lower Fréchet-Hoeffding bound C_* as copula.

Random variables who have C^* or C_* as copula are called *comonotone* or *countermonotone*, resp.

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3. A short review of copulas

Sklar's Theorem: Let H denote some n-dimensional distribution function with marginal distribution functions $F_1,\cdots,F_n$. Then there exists a copula C such that for all real $(x_1,\cdots,x_n)$,

$$H(x_1,\cdots,x_n)=C\left(F_1(x_1),\cdots,F_n(x_n)\right).$$

If all the marginal distribution functions are continuous, then the copula is unique. Moreover, the converse of the above statement is also true. If we denote by $F_1^{-1},\cdots,F_n^{-1}$ the generalized inverses of the marginal distribution functions (or quantile functions), then for every $(u_1,\cdots,u_n)$ in the unit cube,

$$C(u_1,\cdots,u_n)=H\left(F_1^{-1}(u_1),\cdots F_n^{-1}(u_n)\right).$$

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4. Correlation and diversification

Example A. Let the joint distribution of the non-negative risks X and Y be given by the following table (where $\alpha = 0.005$) :

$P(X=x, Y=y)$		$x=0$	$x=50$	$x=100$	$P(Y=y)$	$P(Y\le y)$
y	0	β	$0.440-\beta$	0.000	0.440	0.440
	40	$0.554-\beta$	β	0.001	0.555	0.995
	50	0.000	0.001	0.004	0.005	1.000
	$P(X=x)$	0.554	0.441	0.005		
	$P(X\le x)$	0.554	0.995	1.000		

with $0 \le \beta \le 0.440$, giving $\text{VaR}_\alpha(X) = 50$, $\text{VaR}_\alpha(Y) = 40$.

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4. Correlation and diversification

For the moments of X and Y, we obtain (with σ denoting the standard deviation):

$E(X)$	$E(Y)$	$\sigma(X)$	$\sigma(Y)$	$\rho(\beta)=\rho(X,Y)$
22.550	22.450	25.377	19.912	$-0.9494+3.9579\beta$

which shows that the range of possible risk correlations is the interval $[-0.9494;\,0.7921]$, with a zero correlation being attained for $\beta=0.2399$.

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4. Correlation and diversification

The following table shows the distribution of the aggregated risk $S = X + Y$:

s	0	40	50	90	100	140	150
$P(S=s)$	β	$0.554-\beta$	$0.440-\beta$	β	0.001	0.001	0.004
$P(S\leq s)$	β	0.554	$0.994-\beta$	0.994	0.995	0.996	1.000

giving a risk concentration due to

$$\mathrm{VaR}_\alpha(S) = 100 > 90 = \mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y),$$

independent of the parameter β and hence also independent of the possible correlations between X and Y!

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4. Correlation and diversification

Proposition A. Let X and Y be non-negative risks with cumulative distribution functions F_X and F_Y, resp. which are continuous and strictly increasing on their support. Denote, for a fixed $\alpha \in (0,1)$,

$$Q^*(\alpha,\delta) := \min\{Q_X(u) + Q_Y(2-\alpha-\delta-u) \mid 1-\alpha-\delta \le u \le 1\} \quad \text{for } 0 \le \delta < 1-\alpha.$$

Then there exists a sufficiently small $\varepsilon \in (0,1-\alpha)$ with the property

$$Q^*(\alpha,\varepsilon) > Q_X(1-\alpha) + Q_Y(1-\alpha) = \mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y).$$

Assume further that the random vector (U,V) has a copula C as joint distribution function with the properties

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4. Correlation and diversification

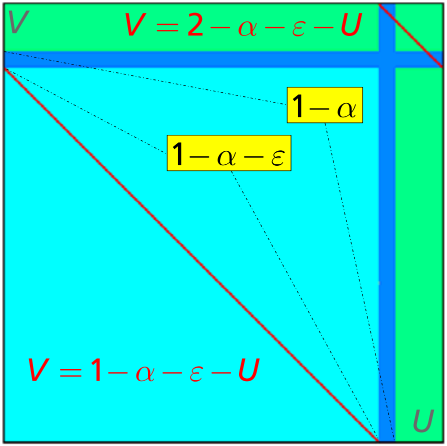
$V < 1-\alpha-\varepsilon$ iff $U < 1-\alpha-\varepsilon$ and $V = 2-\alpha-\varepsilon-U$ iff $U \geq 1-\alpha-\varepsilon$.

If we define $X^* := Q_X(U)$, $Y^* := Q_Y(V)$, $S^* := X^* + Y^*$, then (X^*, Y^*) has the same marginal distributions as (X,Y), and it holds

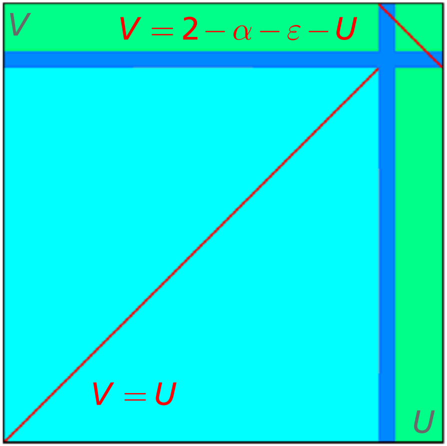
$$\mathrm{VaR}_\alpha(X^*+Y^*) \geq Q^*(\alpha,\varepsilon) > \mathrm{VaR}_\alpha(X^*) + \mathrm{VaR}_\alpha(Y^*) = \mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y),$$

i.e. there is a risk concentration effect. Moreover, the correlation $\rho(X^*, Y^*)$ is minimal if $V = 1-\alpha-\varepsilon-U$ for $U < 1-\alpha-\varepsilon$ (lower extremal copula $\underline{C}$) and maximal if $V = U$ for $U < 1-\alpha-\varepsilon$ (upper extremal copula $\overline{C}$).

4. Correlation and diversification



lower extremal copula $\underline{C}$



upper extremal copula $\overline{C}$

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4. Correlation and diversification

Example B. Assume that the risks X and Y follow the same lognormal distribution $\mathcal{LN}\left(-\frac{\sigma^2}{2},\sigma\right)$ with $\sigma > 0$ which corresponds to $E(X) = E(Y) = 1.$

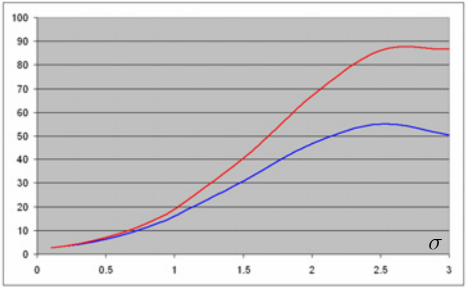
The following table shows some numerical results for the extreme copulas $\underline{C}$ and $\overline{C}$ in Proposition A, especially the maximal range of correlations induced by them. According to the Solvency II standard, we choose $\alpha = 0.005$ (and $\varepsilon = 0.001$, which will be sufficient here).

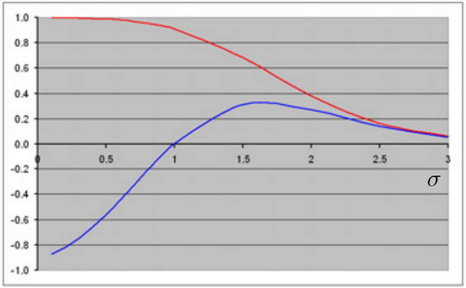
4. Correlation and diversification

σ	0.1	0.2	0.3	0.4	0.5	0.6	0.7
$\mathrm{VaR}_\alpha(X)=\mathrm{VaR}_\alpha(Y)$	1.2873	1.6408	2.0704	2.5866	3.1992	3.9177	4.7497
$\mathrm{VaR}_\alpha(X)+\mathrm{VaR}_\alpha(Y)$	2.5746	3.2816	4.1408	5.1732	6.3984	7.8354	9.4994
$\mathrm{VaR}_\alpha\left(X^*+Y^*\right)$	2.6205	3.3994	4.3661	5.5520	6.9901	8.7134	10.7537
$\rho_{\min}\left(X^*,Y^*\right)$	–0.8719	–0.8212	–0.7503	–0.6620	–0.5598	–0.4480	–0.3310
$\rho_{\max}\left(X^*,Y^*\right)$	0.9976	0.9969	0.9951	0.9920	0.9873	0.9802	0.9700

$$P^X=P^Y=\mathcal{LN}\left(-\frac{\sigma^2}{2},\sigma\right)$$

4. Correlation and diversification

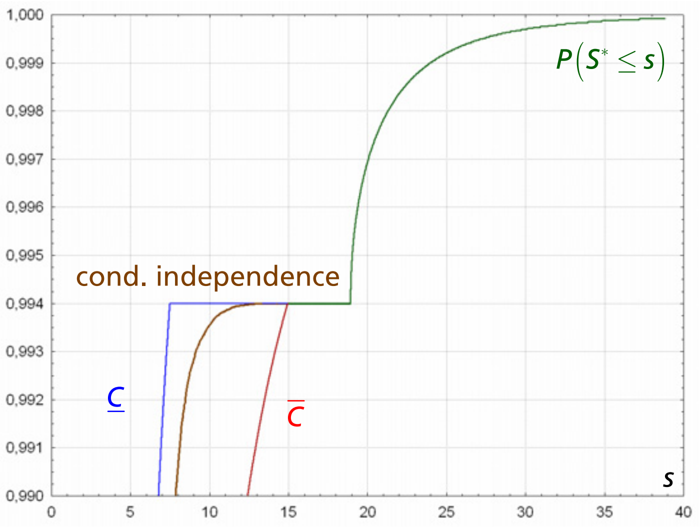


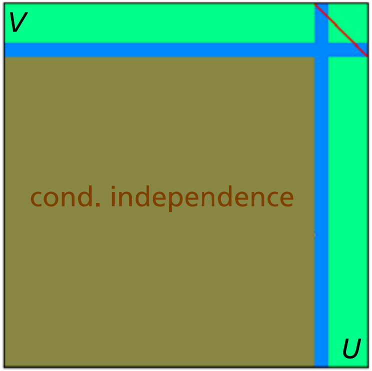


Left: graph of $\mathrm{VaR}_\alpha(X^*+Y^*)$ and $\mathrm{VaR}_\alpha(X)+\mathrm{VaR}_\alpha(Y)$ as functions of σ

Right: graph of $\rho_{\max}(X^*,Y^*)$ and $\rho_{\min}(X^*,Y^*)$ as functions of σ

4. Correlation and diversification





Upper part of the cdf for aggregated risk $S^* = X^* + Y^*$ with $\sigma = 1$

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5. Tail dependence and diversification

Fallacy:

A *positive* tail dependence *reduces* the diversification effect,
a *negative* tail dependence *increases* the diversification effect.

Remark:

The copula construction of Proposition A has no upper tail dependence:

$$\lambda_u = \lim_{u \uparrow 1} \frac{P(U > u, V > u)}{1-u} = 0$$

because for $1-(\alpha+\varepsilon)/2 < u \leq 1$, we have $2-\alpha-\varepsilon-u < u$ and hence

$$P(U > u, V > u) = P(U > u, 2-\alpha-\varepsilon-U > u) = P(u < U < 2-\alpha-\varepsilon-u) = P(\varnothing) = 0.$$

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5. Tail dependence and diversification

Proposition B. Assume that the conditions of Proposition A hold, with the following modification of the copula construction:

$$V < 1-\alpha-\varepsilon \text{ iff } U < 1-\alpha-\varepsilon \text{ and } V = \begin{cases} 2-\alpha-\varepsilon-\gamma-U & \text{if } 1-\alpha-\varepsilon \le U < 1-\gamma \\ U & \text{if } 1-\gamma \le U \le 1 \end{cases}$$

with some non-negative $\gamma < \alpha$. Then, for sufficiently small ε and γ,

$$\min\{Q_X(u)+Q_Y(2-\alpha-\varepsilon-\gamma-u) \mid 1-\alpha-\varepsilon \le u \le 1-\gamma\} > Q_X(1-\alpha)+Q_Y(1-\alpha)$$

and we have still a risk concentration effect:

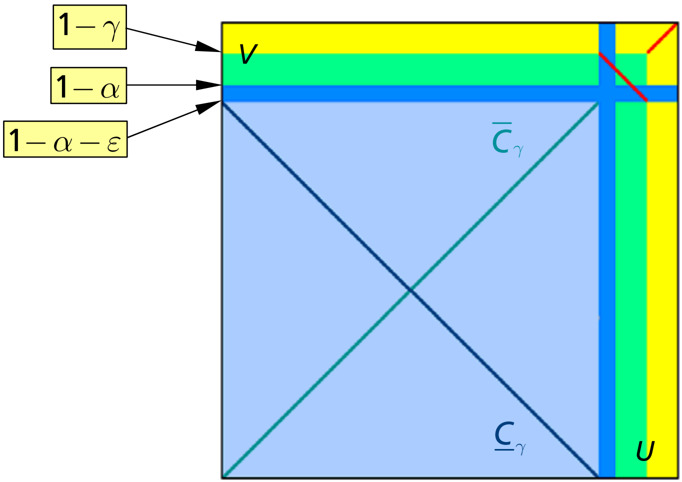
$$\text{VaR}_\alpha\left(X^*+Y^*\right) > \text{VaR}_\alpha(X)+\text{VaR}_\alpha(Y).$$

5. Tail dependence and diversification

Moreover, under this copula construction, the correlation $\rho\left(X^*,Y^*\right)$ is again minimal if $V = 1-\alpha-\varepsilon-U$ for $U < 1-\alpha-\varepsilon$ (lower extremal copula $\underline{C}_\gamma$) and maximal if $V = U$ for $U < 1-\alpha-\varepsilon$ (upper extremal copula $\overline{C}_\gamma$). Further, the risks are in all cases upper tail dependent with

$$\lambda_u = \lim_{u\uparrow 1} \frac{P\left(U > u, V > u\right)}{1-u} = 1.$$

5. Tail dependence and diversification



lower and upper extremal copulas with tail dependence 1

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5. Tail dependence and diversification

Example C. Assume again that the risks X and Y follow the same log-normal distribution $\mathcal{LN}\left(-\frac{\sigma^2}{2},\sigma\right)$ with $\sigma>0$, i.e. $E(X)=E(Y)=1.$

The following table shows some numerical results for the extreme copulas $\underline{C}_\gamma$ and $\overline{C}_\gamma$ in Proposition B, especially the maximal range of correlations induced by them. According to the Solvency II standard, we choose $\alpha=0.005$ (with $\varepsilon=0.001$ and $\gamma=0.0005$ which will be sufficient here).

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5. Tail dependence and diversification

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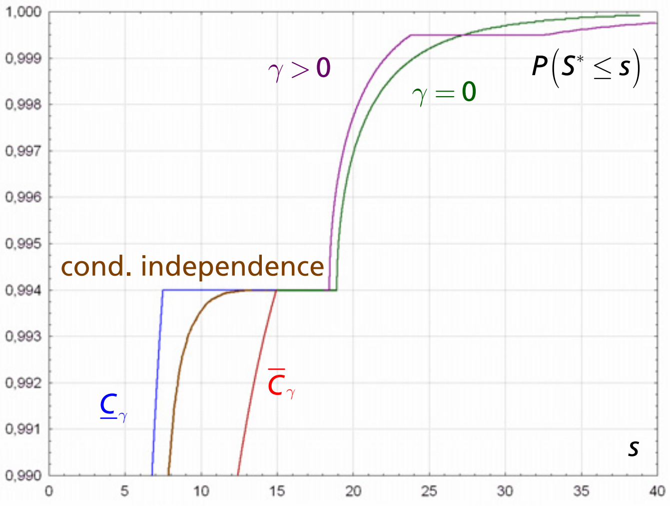
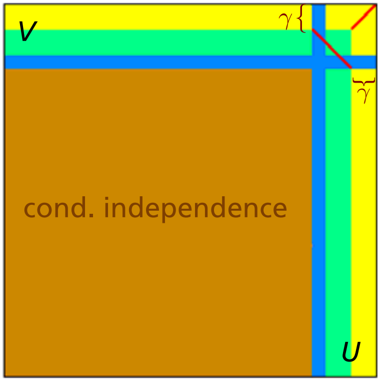
$\mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y)$	2.5746	3.2816	4.1408	5.1732	6.3984	7.8354	9.4994
$\mathrm{VaR}_\alpha\left(X^* + Y^*\right)$	2.6134	3.3811	4.3308	5.4923	6.8962	8.5730	10.5516
$\rho_{\min}\left(X^*, Y^*\right)$	–0.8710	–0.8193	–0.7471	–0.6568	–0.5515	–0.4349	–0.3107
$\rho_{\max}\left(X^*, Y^*\right)$	0.9993	0.9988	0.9981	0.9969	0.9953	0.9929	0.9974

$$P^X = P^Y = \mathcal{LN}\left(-\frac{\sigma^2}{2}, \sigma\right)$$

5. Tail dependence and diversification





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6. Summary

- Neither the notion of correlation nor the notion of tail dependence as such has in general a direct impact on diversification under the risk measure Value at Risk.
- Any attempt to implement such concepts into a simple Pillar One standard model under Solvency II for the purpose of a reduction of the Solvency Capital Requirement in case of a diversification effect cannot be justified by mathematical reasoning.

The concept of diversification is meaningless unless applied in the context of a well-defined joint model. Any interpretation of diversification in the absence of such a model should be avoided.

[modified after McNeil, Frey and Embrechts (2005), p. 205]

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