

Correlation, tail dependence and diversification

Dietmar Pfeifer

University of Oldenburg, Germany

Faculty of Mathematics and Science

Abstract

In this contribution we analyze the interplay between correlation, tail dependence and diversification between risks which has a great impact on the calculation of the Solvency Capital Requirement under the Solvency II directive. The analysis shows that the prevailing assumption of an economic relationship between diversification and correlation or stochastic dependence is misleading under the risk measure VaR (Value at Risk) used under Solvency II and can lead to an underestimation of the necessary economic capital.

1 Introduction

What is frequently abbreviated as “Solvency II” is perhaps the most challenging legislative adventure in the European Union (besides Basel II/III for the banking sector) in the last decade. It is a fundamentally new, risk driven approach towards a harmonization of financial regulation for insurance and reinsurance companies writing business in the European Union. One of the major aims of the Solvency II framework is a customer protection limiting the yearly ruin probability of the company to at most 0,5% by requiring sufficient economic capital. The calculation of this so called Solvency Capital Requirement (SCR) is based on a complicated mathematical and statistical framework derived from an economic balance sheet approach (for more details, see e.g. Buckham, Wahl and Rose (2011), Cruz (2009), Doff (2007) or Sandström (2006)). An essential aspect in the SCR calculation here is the notion of diversification, which aims at a reduction of the overall capital requirement by “distributing” risk in an appropriate way. There are several definitions and explanations of this term, some of which are presented in the sequel.

“Although it is an old idea, the measurement and allocation of diversification in portfolios of asset and / or liability risks is a difficult problem, which has so far found many answers. The diversification effect of a portfolio of risks is the difference between the sum of the risk measures of stand-alone risks in the portfolio and the risk measure of all risks in the portfolio taken together, which is typically non-negative, at least for positive dependent risks.”

[Hürlimann (2009a), p. 325]

“Diversification arises when different activities complement each other, in the field of both return and risk. [...] The diversification effect is calculated by using correlation factors. Correlations are statistical measures assessing the extend to which events could occur simultaneously. [...] A correlation factor of 1 implies that certain events will always occur simultaneously. Hence, there is no diversification effect and two risks identically add up. Risk managers tend to say that such risks are perfectly correlated (i.e., they have a high correlation factor), meaning that these two risks do not actually diversify at all. A correlation factor of 0 implies that diversification effects are present and a certain diversification benefit holds.”

[Doff (2007), p. 167f.]

“By diversifiable we mean that if a risk category can be subdivided into risk classes and the risk charge of the total risk is not higher than the sum of the risk charges of each subrisk, then we have the effect of diversification. [...] This effect can be measured as the difference between the sum of several capital charges and the total capital charge when dependency between them is taken into account.”

[Sandström (2006), p. 188]

“In order to promote good risk management and align regulatory capital requirements with industry practices, the Solvency Capital Requirement should be determined as the economic capital to be held by insurance and reinsurance undertakings in order to ensure that ruin occurs no more often than once in every 200 cases. [...] That economic capital should be calculated on the basis of the true risk profile of those undertakings, taking account of the impact of possible risk-mitigation techniques, as well as diversification effects. [...] ‘Diversification effects’ means the reduction in the risk exposure of insurance and reinsurance undertakings and groups related to the diversification of their business, resulting from the fact that the adverse outcome from one risk can be offset by a more favourable outcome from another risk, where those risks are not fully correlated. The Basic Solvency Capital Requirement shall comprise individual risk modules, which are aggregated [...] The correlation coefficients for the aggregation of the risk modules [...], shall result in an overall Solvency Capital Requirement [...] Where appropriate, diversification effects shall be taken into account in the design of each risk module.”

[Directive 2009/138/EC, (64) p. 7; (37) p. 24; Article 104, p. 52]

One central idea that is common to all of these explanations is that a small, zero or even negative correlation between risks implies a diversification effect, while a large correlation or positive dependence implies the opposite. This is, however, largely based on a naïve understanding of the relationship between correlation and dependence which is not at all justified from a rigorous statistical point of view (see e.g. Mari and Kotz (2001)). This fact has also been emphasized by McNeil, Frey and Embrechts (2005) in Chapters 5 and 6 of their monograph, and in part also by Artzner, Delbaen, Eber and Heath (1999).

A better way to tackle the understanding of a diversification effect is to replace the notion of correlation by the notion of copulas which describe the dependence structure between risks completely (see e.g. Nelsen (2006) for a sophisticated survey). With respect to the “dangerousness” of joint risks, tail dependence is often used as a characteristic quantity (see e.g. McNeil, Frey and Embrechts (2005), Section 5.2.3). In case of a positive upper coefficient of tail dependence, it is likely that extreme events will occur more frequently simultaneously, just in the spirit of Doff’s explanation of diversification above. This might suggest that risks with positive upper tail dependence are less exposed to diversification than those with zero upper tail dependence. However, a more sophisticated analysis shows that this is also not true in general.

The aim of this contribution is twofold:

firstly, to show that the notion of correlation is completely disjoint from the notion of diversification under the risk measure VaR used in the Solvency II directive, i.e. we shall show that a state of no diversification between risks can be achieved with almost arbitrary positive and negative correlation coefficients, especially with the same marginal risk distributions,

and secondly, that a state of no diversification between risks can also be achieved with a zero tail dependence coefficient, or even worse, with a partial countermonotonic dependence structure, in particular for risks being lognormally distributed which is a basic assumption in the Pillar One standard model of Solvency II.

2 A short review of risk measures

In this section, we shall only focus on risk measures for non-negative risks since these are the essential quantities in insurance, and are also the fundamentals of the SCR calculation under Solvency II. We follow a simplified setup as in Sandström (2006), Section 7.4 which is formally slightly different from the approach in Artzner, Delbaen, Eber and Heath (1999) or McNeil, Frey and Embrechts (2005), Chapter 6.

Definition 1. Let \mathfrak{X} be a suitable set of non-negative random variables X on a probability space (Ω, \mathcal{A}, P) . A risk measure R on \mathfrak{X} is a mapping $\mathfrak{X} \rightarrow \mathbb{R}^+$ with the following properties:

$$P^X = P^Y \Rightarrow R(X) = R(Y) \text{ for all } X, Y \in \mathfrak{X}, \quad (1)$$

i.e. the risk measure depends only on the distribution of the risk X ;

$$R(cX) = cR(X) \text{ for all } X \in \mathfrak{X} \text{ and } c \geq 0, \quad (2)$$

i.e. the risk measure is *scale-invariant*;

$$R(X + c) = R(X) + c \text{ for all } X \in \mathfrak{X} \text{ and } c \geq 0, \quad (3)$$

i.e. the risk measure is *translation-invariant*;

$$R(X) \leq R(Y) \text{ for all } X, Y \in \mathfrak{X} \text{ with } X \leq Y, \quad (4)$$

i.e. the risk measure is *monotone*.

The risk measure is called *coherent*, if it additionally has the subadditivity property:

$$R(X + Y) \leq R(X) + R(Y) \text{ for all } X, Y \in \mathfrak{X}. \quad (5)$$

This last property is the crucial point: it induces a diversification effect for *arbitrary* non-negative risks X_1, \dots, X_n (dependent or not) since it follows by induction that coherent risk measures have the property

$$R\left(\sum_{k=1}^n X_k\right) \leq \sum_{k=1}^n R(X_k) \text{ for any } n \in \mathbb{N}. \quad (6)$$

In what follows we shall use the term “(risk) *concentration* effect” as opposite to “*diversification* effect”, characterized by the converse inequality

$$R(X + Y) > R(X) + R(Y) \text{ for some } X, Y \in \mathfrak{X}. \quad (7)$$

Example 1. The popular standard deviation principle *SDP* which is sometimes used for tariffing in insurance is defined as

$$SDP(X) = E(X) + \gamma\sqrt{Var(X)} \text{ for a fixed } \gamma > 0 \text{ and } X \in \mathfrak{X} = \mathfrak{L}_+^2(\Omega, \mathcal{A}, P), \quad (8)$$

the set of non-negative square-integrable random variables on (Ω, \mathcal{A}, P) . Obviously, *SDP* fulfils the properties (1) to (3) and (5); the latter because of

$$\begin{aligned} SDP(X + Y) &= E(X) + E(Y) + \gamma\sqrt{Var(X) + Var(Y) + 2\rho(X, Y)\sqrt{Var(X) \cdot Var(Y)}} \\ &\leq E(X) + E(Y) + \gamma\sqrt{Var(X) + Var(Y) + 2\sqrt{Var(X) \cdot Var(Y)}} \\ &= E(X) + E(Y) + \gamma\sqrt{(\sqrt{Var(X)} + \sqrt{Var(Y)})^2} = SDP(X) + SDP(Y) \end{aligned} \quad (9)$$

for all $X, Y \in \mathfrak{X}$. Here $\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$ denotes the correlation between X and Y .

However, *SDP* does in general not fulfil property (4) and is hence not a proper risk measure, as can be seen as follows: Let Z be a random variable binomially distributed over $\{0, 1\}$ with

$P(Z = 1) = 1 - P(Z = 0) = p$, and $\frac{1}{1 + \gamma^2} < p < 1$. Consider $X := 2Z$ and $Y := 1 + Z$. Then

$X \leq Y$, but $R(X) = 2p + 2\gamma\sqrt{p(1-p)} > 1 + p + \gamma\sqrt{p(1-p)} = R(Y)$.

Example 2. The risk measure used in Basel II/III and Solvency II is the Value-at-Risk VaR, being defined as a (typically high) quantile of the risk distribution:

$$VaR_\alpha(X) := Q_X(1 - \alpha) \text{ for } X \in \mathfrak{X} \text{ and } 0 < \alpha < 1, \quad (10)$$

where Q_X denotes the quantile function

$$Q_X(u) := \inf \{x \in \mathbb{R} \mid P(X \leq x) \geq u\} \text{ for } 0 < u < 1. \quad (11)$$

Value-at-Risk is a proper risk measure, but not coherent in general. This topic will be discussed in more detail in the next section (for a more general discussion, see e.g. McNeil, Frey and Embrechts (2005), Section 6.1.2).

The “smallest” coherent risk measure above VaR is the expected shortfall ES, which is in general defined as

$$ES_\alpha(X) := \frac{1}{\alpha} \left\{ E \left(X \cdot \mathbb{1}_{\{X \geq \text{VaR}_\alpha(X)\}} \right) + \text{VaR}_\alpha(X) [\alpha - P(X \geq \text{VaR}_\alpha(X))] \right\} \quad (12)$$

for $X \in \mathfrak{X}$ and $0 < \alpha < 1$, where $\mathbb{1}_A$ denotes the indicator random variable of some event (measurable set) A . In case that $P(X \geq \text{VaR}_\alpha(X)) = \alpha$, this formula simplifies to

$$ES_\alpha(X) = E(X \mid X \geq \text{VaR}_\alpha(X)) = \frac{1}{\alpha} \int_0^\alpha \text{VaR}_u(X) du \quad (13)$$

(see McNeil, Frey and Embrechts (2005), Definition 2.15 and Remark 2.17; note that the role of α and $1 - \alpha$ are interchanged there). A more thorough discussion on the relationship between VaR and ES (and other coherent risk measures) in connexion with Wang’s distortion measures can be found in Hürlimann (2004).

Expected shortfall is the risk measure that is used in the Swiss Solvency Test (SST), see e.g. Sandström (2006), Section 6.8 or Cruz (2009), Chapter 17.

3 A short review of copulas

The copula approach allows for a separate treatment of the margins of joint risks and the dependence structure between them. The name “copula” goes back to Abe Sklar in 1959 who used it as a function which couples a joint distribution function with its univariate margins. For an extensive survey, see e.g. Nelsen (2006).

Definition 2. A copula (in n dimensions) is a function C defined on the unit cube $[0,1]^n$ with the following properties:

1. the range of C is the unit interval $[0,1]$;
2. $C(\mathbf{u})$ is zero for all $\mathbf{u} = (u_1, \dots, u_n)$ in $[0,1]^n$ for which at least one coordinate is zero;
3. $C(\mathbf{u}) = u_k$ if all coordinates of \mathbf{u} are 1 except the k -th one;
4. C is n -increasing in the sense that for every $\mathbf{a} \leq \mathbf{b}$ in $[0,1]^n$ the volume assigned by C to the subinterval $[\mathbf{a}, \mathbf{b}] = [a_1, b_1] \times \dots \times [a_n, b_n]$ is nonnegative.

A copula can alternatively be characterized as a multivariate distribution function with univariate marginal distribution functions that belong to a continuous uniform distribution over the unit interval $[0,1]$.

It can be shown that every copula is bounded by the so-called Fréchet-Hoeffding bounds, i.e.

$$C_*(\mathbf{u}) := \max(u_1 + \dots + u_n - n + 1, 0) \leq C(u_1, \dots, u_n) \leq C^*(\mathbf{u}) := \min(u_1, \dots, u_n). \quad (14)$$

The upper Fréchet-Hoeffding bound C^* is a copula itself for any dimension; however, the lower Fréchet-Hoeffding bound C_* is a copula in two dimensions only. If X is any real random variable, then the random vector $\mathbf{X} = (X, X, \dots, X)$ with n components possesses the upper Fréchet-Hoeffding bound C^* as copula, while the random vector $\mathbf{X} = (X, -X)$ with two components possesses the lower Fréchet-Hoeffding bound C_* as copula. Random variables who have C^* or C_* , resp. as copula are also called *comonotone* or *countermonotone*, resp.

An important and well-studied copula is the independence copula, given by $C(\mathbf{u}) = \prod_{i=1}^n u_i$.

The following theorem due to Sklar justifies the role of copulas as dependence functions:

Proposition 1. Let H denote some n -dimensional distribution function with marginal distribution functions F_1, \dots, F_n . Then there exists a copula C such that for all real (x_1, \dots, x_n) ,

$$H(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)).$$

If all the marginal distribution functions are continuous, then the copula is unique. Moreover, the converse of the above statement is also true. If we denote by $F_1^{-1}, \dots, F_n^{-1}$ the generalized inverses of the marginal distribution functions (or quantile functions), then for every (u_1, \dots, u_n) in the unit cube,

$$C(u_1, \dots, u_n) = H(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)).$$

For a proof, see Nelsen (2006), Theorem 2.10.9 and the references given therein.

The above theorem shows that copulas remain invariant under strictly monotone transformations of the same kind of the underlying random variables (either increasing or decreasing).

The relationship between correlation and copulas shows the following result.

Proposition 2. Let (X, Y) be a bivariate random vector with a copula C and marginal distribution functions F and G such that $E(|X|) < \infty$, $E(|Y|) < \infty$ and $E(|XY|) < \infty$.

Then the covariance between X and Y can be expressed in the following way:

$$\text{Cov}(X, Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [C(F(x), G(y)) - F(x)G(y)] dx dy \quad (15)$$

For a proof see e.g. McNeil, Frey and Embrechts (2005), Lemma 5.24.

4 Correlation and diversification

Before going into deeper details, we start with an illustrative example showing essentially that risk concentration under VaR can occur with almost all positive and negative correlation coefficients, even if the marginal distributions remain fixed. According to the Solvency II standard, we choose $\alpha = 0.005$ for simplicity here, but the example can be reformulated accordingly with any other value of $0 < \alpha < 1$.

Example 3. Let the joint distribution of the non-negative risks X and Y be given by the following table:

$P(X = x, Y = y)$		x			$P(Y = y)$	$P(Y \leq y)$
		0	50	100		
y	0	β	$0.440 - \beta$	0.000	0.440	0.440
	40	$0.554 - \beta$	β	0.001	0.555	0.995
	50	0.000	0.001	0.004	0.005	1.000
$P(X = x)$		0.554	0.441	0.005		
$P(X \leq x)$		0.554	0.995	1.000		

Tab. 1

with $0 \leq \beta \leq 0.440$, giving $\text{VaR}_\alpha(X) = 50$, $\text{VaR}_\alpha(Y) = 40$. For the moments of X and Y , we obtain (with σ denoting the standard deviation):

$E(X)$	$E(Y)$	$\sigma(X)$	$\sigma(Y)$	$\rho(\beta) = \rho(X, Y)$
22.550	22.450	25.377	19.912	$-0.9494 + 3.9579\beta$

Tab. 2

This shows that the range of possible risk correlations is the interval $[-0.9494; 0.7921]$, with a zero correlation being attained for $\beta = 0.2399$.

The following table shows the distribution of the aggregated risk $S = X + Y$:

s	0	40	50	90	100	140	150
$P(S = s)$	β	$0.554 - \beta$	$0.440 - \beta$	β	0.001	0.001	0.004
$P(S \leq s)$	β	0.554	$0.994 - \beta$	0.994	0.995	0.996	1.000

Tab. 3

We thus obtain a risk concentration due to $\text{VaR}_\alpha(S) = 100 > 90 = \text{VaR}_\alpha(X) + \text{VaR}_\alpha(Y)$, independent of the parameter β and hence also independent of the possible correlations between X and Y .

A closer look to the joint distribution of X and Y shows that the reason for this perhaps unexpected result is the fact that although one can have a “diversification effect” in the central body of the distribution, where a fraction of little less than $1 - \alpha$ of the risk pairs are located, the essential “concentration effect”, however, is caused by a joint occurrence of very high losses, with a fraction of α of all risk pairs.

The following result is related to the consideration of “worst VaR scenarios” as in McNeil, Frey and Embrechts (2005), Section 6.2.

Proposition 3. Let X and Y be non-negative risks with cumulative distribution functions F_X and F_Y , resp. which are continuous and strictly increasing on their support. Denote, for a fixed $\alpha \in (0, 1)$,

$$Q^*(\alpha, \delta) := \min \{Q_X(u) + Q_Y(2 - \alpha - \delta - u) \mid 1 - \alpha - \delta \leq u \leq 1\} \text{ for } 0 \leq \delta < 1 - \alpha. \quad (16)$$

Then there exists a sufficiently small $\varepsilon \in (0, 1 - \alpha)$ with the property

$$Q^*(\alpha, \varepsilon) > Q_X(1 - \alpha) + Q_Y(1 - \alpha) = \text{VaR}_\alpha(X) + \text{VaR}_\alpha(Y). \quad (17)$$

Assume further that the random vector (U, V) has a copula C as joint distribution function with the properties

$$V < 1 - \alpha - \varepsilon \text{ iff } U < 1 - \alpha - \varepsilon \text{ and } V = 2 - \alpha - \varepsilon - U \text{ iff } U \geq 1 - \alpha - \varepsilon. \quad (18)$$

If we define

$$X^* := Q_X(U), Y^* := Q_Y(V), S^* := X^* + Y^*, \quad (19)$$

then the random vector (X^*, Y^*) has the same marginal distributions as (X, Y) , and it holds

$$\text{VaR}_\alpha(X^* + Y^*) \geq Q^*(\alpha, \varepsilon) > \text{VaR}_\alpha(X^*) + \text{VaR}_\alpha(Y^*) = \text{VaR}_\alpha(X) + \text{VaR}_\alpha(Y), \quad (20)$$

i.e. there is a risk concentration effect. Moreover, under the assumption (18), the correlation $\rho(X^*, Y^*)$ is minimal if $V = 1 - \alpha - \varepsilon - U$ for $U < 1 - \alpha - \varepsilon$ (lower extremal copula \underline{C}) and maximal if $V = U$ for $U < 1 - \alpha - \varepsilon$ (upper extremal copula \overline{C}).

Proof. By assumption, the (non-negative) quantile functions Q_X and Q_Y are continuous and strictly increasing over the interval $[0, 1]$ (with a possibly infinite value at the point 1), so that

$$Q^*(\alpha, 0) = \min\{Q_X(u) + Q_Y(2 - \alpha - u) \mid 1 - \alpha \leq u \leq 1\} > Q_X(1 - \alpha) + Q_Y(1 - \alpha), \quad (21)$$

the minimum being actually attained. Since by the continuity assumptions above, $Q^*(\alpha, \varepsilon)$ is continuous in ε and decreasing when ε is increasing, relation (17) follows.

The copula construction above now implies that

$$P(S^* \leq s) \leq 1 - \alpha - \varepsilon \text{ for } s \leq Q_X(1 - \alpha - \varepsilon) + Q_Y(1 - \alpha - \varepsilon) = \text{VaR}_{\alpha+\varepsilon}(X) + \text{VaR}_{\alpha+\varepsilon}(Y) \quad (22)$$

$$P(S^* \leq s) = 1 - \alpha - \varepsilon \text{ for } \text{VaR}_{\alpha+\varepsilon}(X) + \text{VaR}_{\alpha+\varepsilon}(Y) < s < Q^*(\alpha, \varepsilon) \quad (23)$$

$$P(S^* \leq s) \geq 1 - \alpha - \varepsilon \text{ for } s \geq Q^*(\alpha, \varepsilon) \quad (24)$$

Relation (24) in turn implies that $\text{VaR}_\alpha(S^*) = \text{VaR}_\alpha(X^* + Y^*) \geq Q^*(\alpha, \varepsilon)$ which proves the first part of Proposition 3, due to relation (17).

The remainder part follows from Theorem 5.25 in McNeil, Frey and Embrechts (2005) when looking at the conditional distribution of (X^*, Y^*) given the event $\{U < 1 - \alpha - \varepsilon\}$. ■

Note that both types of copulas that provide the extreme values for the correlations, \underline{C} and \overline{C} , are of the type “shuffles of M ”, see Nelsen (2006), Section 3.2.3.

In the following example, we shall show some consequences of Proposition 3 in the case of lognormally distributed risks, which are of special importance for Pillar One under Solvency II, see e.g. Hürliman (2009a, 2009b).

Example 4. To keep things simple and comparable with Solvency II specifications, we shall assume that X and Y follow the same lognormal distribution $\mathcal{LN}(\mu, \sigma)$ with $\mu \in \mathbb{R}$, $\sigma > 0$ and $E(X) = E(Y) = 1$ which corresponds to the case $\mu = -\frac{\sigma^2}{2}$. The following table shows all relevant numerical results for the extreme copulas \underline{C} and \overline{C} in Proposition 3, especially the maximal range of correlations induced by them. According to the Solvency II standard, we choose $\alpha = 0.005$ (and $\varepsilon = 0.001$, which will be sufficient here).

σ	0.1	0.2	0.3	0.4	0.5	0.6	0.7
$\text{VaR}_\alpha(X) = \text{VaR}_\alpha(Y)$	1.2873	1.6408	2.0704	2.5866	3.1992	3.9177	4.7497
$\text{VaR}_\alpha(X) + \text{VaR}_\alpha(Y)$	2.5746	3.2816	4.1408	5.1732	6.3984	7.8354	9.4994
$\text{VaR}_\alpha(X^* + Y^*)$	2.6205	3.3994	4.3661	5.5520	6.9901	8.7134	10.7537
$\rho_{\min}(X^*, Y^*)$	-0.8719	-0.8212	-0.7503	-0.6620	-0.5598	-0.4480	-0.3310
$\rho_{\max}(X^*, Y^*)$	0.9976	0.9969	0.9951	0.9920	0.9873	0.9802	0.9700

σ	0.8	0.9	1.0	1.5	2.0	2.5	3.0
$\text{VaR}_\alpha(X) = \text{VaR}_\alpha(Y)$	5.7011	6.7750	7.9712	15.4675	23.3748	27.5107	25.2162
$\text{VaR}_\alpha(X) + \text{VaR}_\alpha(Y)$	11.4022	13.5500	15.9424	30.9350	46.7496	55.0214	50.4324
$\text{VaR}_\alpha(X^* + Y^*)$	13.1401	15.8969	19.0412	40.4257	66.8923	86.2673	86.7034
$\rho_{\min}(X^*, Y^*)$	-0.2136	-0.1002	0.0050	0.3127	0.2723	0.1399	0.0565
$\rho_{\max}(X^*, Y^*)$	0.9556	0.9362	0.9108	0.6839	0.3794	0.1637	0.0611

Tab. 4

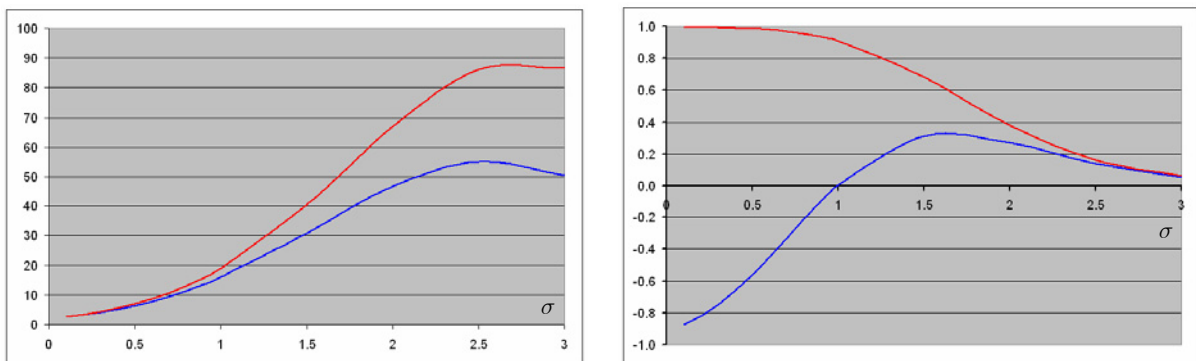


Fig. 1

Left: graph of $\text{VaR}_\alpha(X^* + Y^*)$ [top] and $\text{VaR}_\alpha(X) + \text{VaR}_\alpha(Y)$ [bottom] as functions of σ

Right: graph of $\rho_{\max}(X^*, Y^*)$ [top] and $\rho_{\min}(X^*, Y^*)$ [bottom] as functions of σ

Note that the graph on the right hand side resembles the graph in Fig. 5.8 in McNeil, Frey and Embrechts (2005).

The following graph shows parts of the two cumulative distribution functions under the extreme copulas \underline{C} and \overline{C} for $S^* := X^* + Y^*$ in the case $\sigma = 1$.

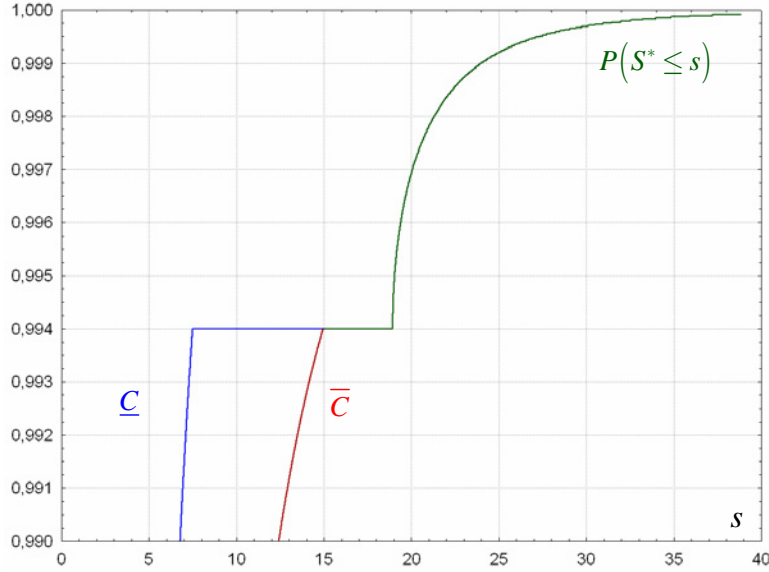


Fig. 2

Note that especially for smaller values of σ (which is typical for the calculation of the SCR in the non-life risk module of Solvency II) the range of possible negative and positive correlations between the risks is quite large, with the same significant discrepancy between the Value at Risk of the aggregated risks and the sum of individual Values at Risk.

Note also that any correlation value ρ of $\rho(X^*, Y^*)$ between $\rho_{\min}(X^*, Y^*)$ and $\rho_{\max}(X^*, Y^*)$ can be achieved by a proper mixture of the extreme copulas \underline{C} and \bar{C} , namely for the copula

$$C(\rho) = \lambda \underline{C} + (1 - \lambda) \bar{C} \quad \text{with} \quad \lambda = \frac{\rho_{\max}(X^*, Y^*) - \rho}{\rho_{\max}(X^*, Y^*) - \rho_{\min}(X^*, Y^*)}. \quad (25)$$

This is a direct consequence from Proposition 2, for example.

There are, of course, also other possibilities to achieve appropriate intermediary values for the correlation, for instance if U and V are conditionally independent given the event $\{U < 1 - \alpha - \varepsilon\}$. The following graph adds a part of the cumulative distribution function of $S^* := X^* + Y^*$ for this case to the graph in Fig. 2. The correlation between X^* and Y^* is here given by $\rho(X^*, Y^*) = 0.3132$.

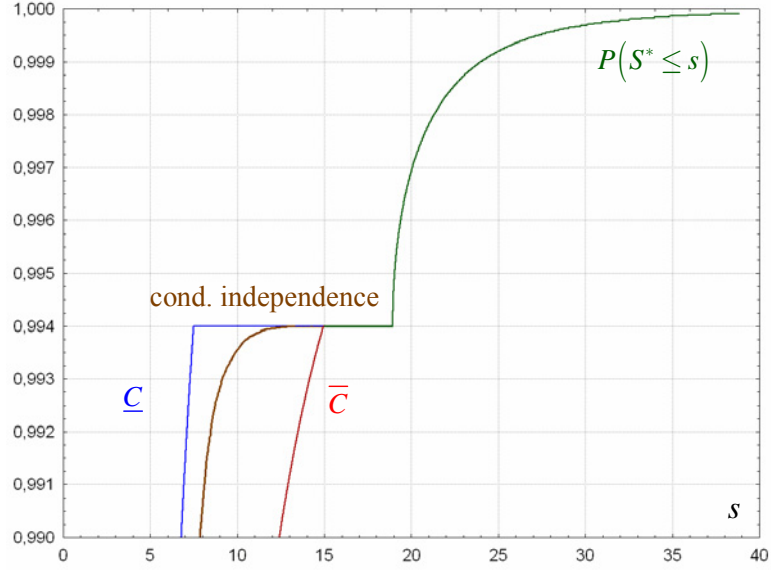


Fig. 3

5 Tail dependence and diversification

As in the case of a large positive correlation between risks, it might be intuitively tempting to assume that a positive upper tail dependence would have a positive impact on risk concentration, too. But this is not true here either. In this section we shall show that a risk concentration effect can occur with and without tail dependence, while the marginal distributions remain unchanged.

Note first that the copula construction of Proposition 3 implies no upper tail dependence since, by the continuity assumption for the marginal distributions made there (see e.g. McNeil, Frey and Embrechts (2005), section 5.2.3),

$$\lambda_u = \lim_{u \uparrow 1} \frac{P(U > u, V > u)}{1 - u} = 0 \quad (26)$$

because for $1 - \frac{\alpha + \varepsilon}{2} < u \leq 1$, we have $2 - \alpha - \varepsilon - u < u$ and hence, for these u ,

$$P(U > u, V > u) = P(U > u, 2 - \alpha - \varepsilon - U > u) = P(u < U < 2 - \alpha - \varepsilon - u) = P(\emptyset) = 0. \quad (27)$$

The following proposition shows that we can incorporate an upper tail dependence in the construction of Proposition 3 without essentially losing the central result.

Proposition 4. Assume that the conditions of Proposition 3 hold, with the following modification of the copula construction in (18):

$$V < 1 - \alpha - \varepsilon \text{ iff } U < 1 - \alpha - \varepsilon \text{ and } V = \begin{cases} 2 - \alpha - \varepsilon - \gamma - U & \text{if } 1 - \alpha - \varepsilon \leq U < 1 - \gamma \\ U & \text{if } 1 - \gamma \leq U \leq 1 \end{cases} \quad (28)$$

with some non-negative $\gamma < \alpha$. Then, for sufficiently small ε and γ , we still have

$$\min \{Q_X(u) + Q_Y(2 - \alpha - \varepsilon - \gamma - u) \mid 1 - \alpha - \varepsilon \leq u \leq 1 - \gamma\} > Q_X(1 - \alpha) + Q_Y(1 - \alpha) \quad (29)$$

and hence again a risk concentration effect, i.e. $\text{VaR}_\alpha(X^* + Y^*) > \text{VaR}_\alpha(X) + \text{VaR}_\alpha(Y)$.

Moreover, under this copula construction, the correlation $\rho(X^*, Y^*)$ is again minimal if $V = 1 - \alpha - \varepsilon - U$ for $U < 1 - \alpha - \varepsilon$ (lower extremal copula \underline{C}_γ) and maximal if $V = U$ for $U < 1 - \alpha - \varepsilon$ (upper extremal copula \overline{C}_γ). Further, the risks are in all cases upper tail dependent with

$$\lambda_u = \lim_{u \uparrow 1} \frac{P(U > u, V > u)}{1 - u} = 1. \quad (29)$$

Proof. The first two parts follow along the lines of the proof of Proposition 3. For the last part, observe that we have $U = V$ iff $1 - \gamma \leq U \leq 1$ which implies $P(U > u, V > u) = 1 - u$ for $1 - \gamma \leq u \leq 1$ and hence (29). ■

Example 5. If we choose $\gamma = 0.0005$, we get the following extension of the results in Example 4, under the same initial conditions:

σ	0.1	0.2	0.3	0.4	0.5	0.6	0.7
$\text{VaR}_\alpha(X) = \text{VaR}_\alpha(Y)$	1.2873	1.6408	2.0704	2.5866	3.1992	3.9177	4.7497
$\text{VaR}_\alpha(X) + \text{VaR}_\alpha(Y)$	2.5746	3.2816	4.1408	5.1732	6.3984	7.8354	9.4994
$\text{VaR}_\alpha(X^* + Y^*)$	2.6134	3.3811	4.3308	5.4923	6.8962	8.5730	10.5516
$\rho_{\min}(X^*, Y^*)$	-0.8710	-0.8193	-0.7471	-0.6568	-0.5515	-0.4349	-0.3107
$\rho_{\max}(X^*, Y^*)$	0.9993	0.9988	0.9981	0.9969	0.9953	0.9929	0.9974

σ	0.8	0.9	1.0	1.5	2.0	2.5	3.0
$\text{VaR}_\alpha(X) = \text{VaR}_\alpha(Y)$	5.7011	6.7750	7.9712	15.4675	23.3748	27.5107	25.2162
$\text{VaR}_\alpha(X) + \text{VaR}_\alpha(Y)$	11.4022	13.5500	15.9424	30.9350	46.7496	55.0214	50.4324
$\text{VaR}_\alpha(X^* + Y^*)$	12.8581	15.5133	18.5310	38.8061	63.3300	80.5429	79.8272
$\rho_{\min}(X^*, Y^*)$	-0.1830	-0.0553	0.0691	0.5658	0.8154	0.9185	0.9636
$\rho_{\max}(X^*, Y^*)$	0.9964	0.9951	0.9744	0.9366	0.9224	0.9423	0.9909

Tab. 5

The following graph shows an extension of the graph in Fig. 3 for the extreme copulas \underline{C}_γ and \overline{C}_γ for $S^* := X^* + Y^*$ in the case $\sigma = 1$. The case $\gamma > 0$ corresponds to upper tail dependence, the case $\gamma = 0$ correspond to the former situation with no upper tail dependence.

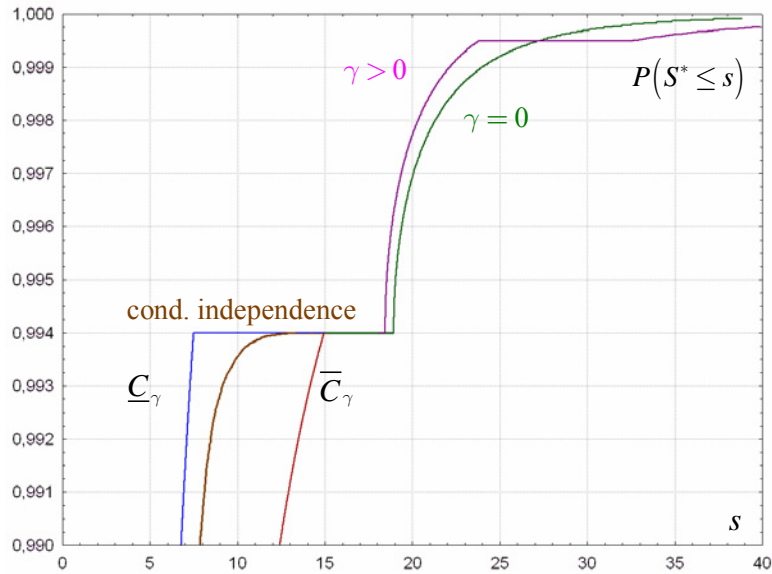


Fig. 4

6 Summary. As the preceding analysis has shown, neither the notion of correlation nor the notion of tail dependence as such has in general a direct impact on diversification under the risk measure Value at Risk. This means that any attempt to implement such concepts into a simple Pillar One standard model under Solvency II for the purpose of a reduction of the Solvency Capital Requirement in case of a diversification effect cannot be justified by mathematical reasoning. We can perhaps summarize the consequences of this insight in a slight modification of a statement in McNeil, Frey and Embrechts (2005), p. 205:

The concept of diversification is meaningless unless applied in the context of a well-defined joint model. Any interpretation of diversification in the absence of such a model should be avoided.

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