Combining Chain Ladder Claims Reserving with Fuzzy Numbers

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Agenda

Motivation

- Fuzzy set theory has been introduced by Zadeh in 1965
- Methodology to cope with linguistic inaccuracies and imprecision
- Models vagueness, not randomness in a stochastic sense
- Various purely computational and stochastic models exist
- Sometimes development factors are adjusted retrospectively
- We propose a model in which no later adjustment is needed and CL factors include vagueness

Literature

- 1980's: First applications of fuzzy methods in insurance
- Most publications of fuzzy methods in claims reserving apply fuzzy regression
- Used e.g. for Taylor's separation method, Sherman's scheme

Classical Chain-Ladder

- Let $C_{i,j}$ cumulative claims made in relative accident year $i \in \{0, \ldots, I\}$ and relative development year $j \in \{0, \ldots, J\}$
- Set of observations $\mathcal{D}_I = \{C_{i,j} \mid i + j \leq I\}$

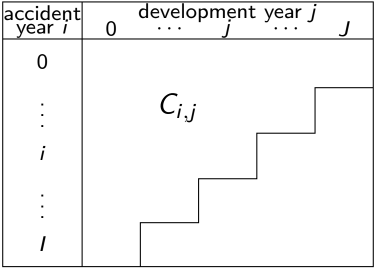


Figure: Development triangle at time $t = I$

Classical Chain-Ladder

Model Assumptions (Distribution-free chain-ladder)

a) *Cumulative claims $C_{i,j}$ of different accident years i are independent.*

b) *There exist development factors $f_0, \ldots, f_{J-1} \in \mathbb{R}^+$ and parameters $\sigma_0^2, \ldots, \sigma_{J-1}^2$ such that*

$$E\left[C_{i,j+1} \mid C_{i,j}\right] = f_j C_{i,j}$$
$$\mathrm{Var}(C_{i,j+1} \mid C_{i,j}) = \sigma_j^2 C_{i,j}$$

holds true for all $i \in \{0, \ldots, I\}$ and $j \in \{0, \ldots, J-1\}$.

Estimation of CL-factors by

$$\hat{f}_j = \frac{\sum_{i=0}^{I-j-1} C_{i,j+1}}{\sum_{i=0}^{I-j-1} C_{i,j}}, \quad j = 0, \ldots, J-1.$$

Example

Drivers under 25

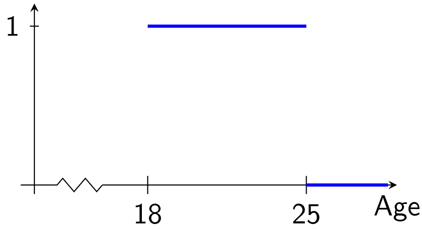


Figure: Classical approach

Young Drivers

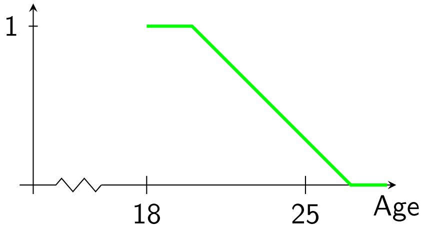


Figure: Fuzzy approach

Fuzzy Sets

Definition (Fuzzy sets)

A fuzzy set $\tilde{A}$ over $\mathbb{R}$ is defined as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in \mathbb{R}\}$$

with membership function $\mu_{\tilde{A}}$ given by:

$$\mu_{\tilde{A}} : \mathbb{R} \to [0, 1]$$

Fuzzy Numbers

Definition (Triangular fuzzy number)

A triangular fuzzy number (TFN) $\tilde{A}$ is a fuzzy set over $\mathbb{R}$ with membership function $\mu_{\tilde{A}}$ given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a+l_a}{l_a} & \text{if } a - l_a \leq x \leq a \\ \frac{a+r_a-x}{r_a} & \text{if } a < x \leq a + r_a \\ 0 & \text{otherwise} \end{cases}$$

for all $a \in \mathbb{R}$ and $l_a, r_a > 0$.

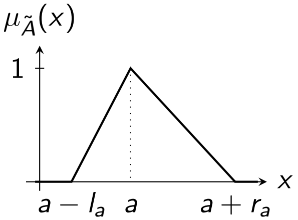


Figure: TFN $\tilde{A} = (a, l_a, r_a)$.

Fuzzy Arithmetic

Definition (Fuzzy arithmetic)

Let $\tilde{A} = (a, l_a, r_a)$ and $\tilde{B} = (b, l_b, r_b)$. The sum, difference and product over $\mathbb{R}$ are defined as

$$\tilde{A} \oplus \tilde{B} = (a, l_a, r_a) \oplus (b, l_b, r_b) = (a + b, l_a + l_b, r_a + r_b)$$
$$\tilde{A} \ominus \tilde{B} = (a, l_a, r_a) \ominus (b, l_b, r_b) = (a - b, l_a + r_b, r_a + l_b)$$
$$\begin{aligned}\tilde{A} \otimes \tilde{B} &= (a, l_a, r_a) \otimes (b, l_b, r_b) \\ &= (ab, al_b + bl_a - l_a l_b, ar_b + br_a + r_a r_b)\end{aligned}$$

for all $a, b, l_a, r_a, l_b, r_b > 0$ with $a > l_a$ and $b > l_b$.

Fuzzy Product

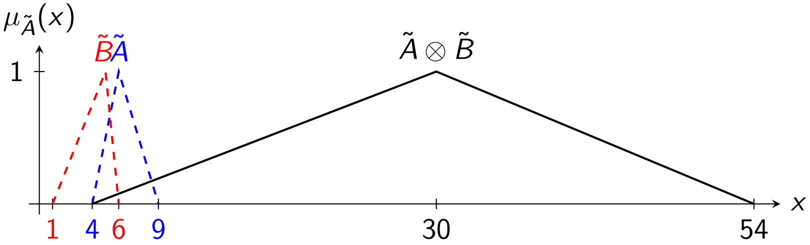


Figure: Fuzzy-product of the TFNs $\tilde{A} = (6, 2, 3)$ and $\tilde{B} = (5, 4, 1)$.

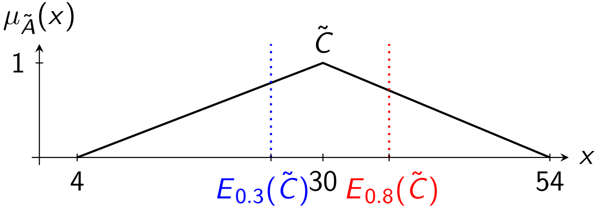
Definition (Expected value of a TFN)

Let $\tilde{A} = (a, l_a, r_a)$ and $\tilde{B}$ be TFNs and $0 \leq \beta \leq 1$. The expected value of the TFN $E_\beta(\tilde{A})$ is given by:

$$E_\beta(\tilde{A}) = a - \frac{1-\beta}{2} l_a + \frac{\beta}{2} r_a$$

In the "conditional expected value" $E_\beta(\tilde{A} \mid \tilde{B})$ the TFN $\tilde{B}$ shall be considered as $\tilde{B} = (B, 0, 0)$, i.e. a crisp number.

Parameter β: "decision-maker risk parameter"

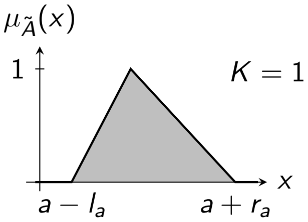


Definition (Uncertainty of a TFN)

Let $\tilde{A} = (a, l_a, r_a)$ and $\tilde{B}$ be TFNs and $K \in \mathbb{R}^+$. The uncertainty of a TFN $Unc_K(\tilde{A})$ is given by:

$$Unc_K(\tilde{A}) = \frac{1}{2}K(l_a + r_a)$$

In the "conditional uncertainty" $Unc_K(\tilde{A} \mid \tilde{B})$ the TFN $\tilde{B}$ shall be considered as a crisp number, i.e. $\tilde{B} = (B, 0, 0)$.



The Model

Model Assumptions (Fuzzy chain-ladder (FCL) model)

- *There exist TFNs* $\tilde{f}_j$ $(j = 0, \ldots, J-1)$ *such that the cumulative claims can be written as*

$$\tilde{C}_{i,j+1} = \tilde{f}_j \otimes \tilde{C}_{i,j}$$

  for all $i = 0, \ldots, I$ *and* $j = 0, \ldots, J-1$.
- *Incremental claims* $X_{i,j+1} = C_{i,j+1} - C_{i,j}$ *are non-negative.*

Lemma

For all $0 \le i \le I$ *and* $0 \le j \le J-1$ *let* $\tilde{C}_{i,j}$ *and* $\tilde{f}_j = (f_j, l_{f_j}, r_{f_j})$ *be positive fuzzy numbers. We have*

$$E_\beta(\tilde{C}_{i,j+1} \mid \tilde{C}_{i,j}) = C_{i,j}\left(f_j - \frac{1-\beta}{2} l_{f_j} + \frac{\beta}{2} r_{f_j}\right)$$

and

$$Unc_K(\tilde{C}_{i,j+1} \mid \tilde{C}_{i,j}) = \tfrac{1}{2}K(l_{f_j} + r_{f_j})C_{i,j}.$$

Goal: Predict ultimate claims by $\hat{\tilde{C}}_{i,J} = C_{i,I-i} \otimes \bigotimes_{j=I-i}^{J-1} \hat{\tilde{f}}_j$

Estimation

Estimator (FCL estimator for the TFNs $\tilde{f}_j$)

The TFNs $\tilde{f}_j = (f_j, l_{f_j}, r_{f_j})$, $j = 0, \dots, J-1$ *are estimated by* $\hat{\tilde{f}}_j = (\hat{f}_j, \hat{l}_{\hat{f}_j}, \hat{r}_{\hat{f}_j})$ *where*

$$\hat{f}_j = \frac{\sum_{i=0}^{I-j-1} C_{i,j+1}}{\sum_{i=0}^{I-j-1} C_{i,j}}$$

$$\hat{l}_{\hat{f}_j} = \hat{r}_{\hat{f}_j} = \frac{\sum_{i=0}^{I-j-1} X_{i,j+1}}{\sum_{i=0}^{I-j-1} C_{i,j}}$$

and

$$X_{i,j+1} = C_{i,j+1} - C_{i,j}$$

for $i = 0, \dots, I$ *and* $j = 0, \dots, J-1$.

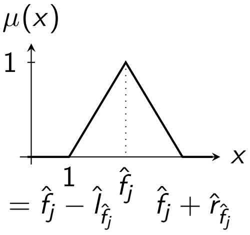


Figure: Example of a FCL estimator.

Reserves

- Claims reserve $\tilde{R}_i$ for a fixed accident year $i = 1, \ldots, I$ is given by

$$\tilde{R}_i = \tilde{C}_{i,J} \ominus \tilde{C}_{i,I-i}$$

- We propose a predictor by

$$\widehat{\tilde{R}}_i = \widehat{\tilde{C}}_{i,J} \ominus \tilde{C}_{i,I-i} = \tilde{C}_{i,I-i} \otimes \left(\bigotimes_{j=I-i}^{J-1} \hat{\tilde{f}}_j - 1 \right)$$

Aggregated reserves

- Aggregated claims reserve is given by

$$\tilde{R} = \bigoplus_{i=1}^{I} \tilde{R}_i = \bigoplus_{i=1}^{I} \left(\tilde{C}_{i,J} \ominus \tilde{C}_{i,I-i} \right)$$

- A predictor is yielded by

$$\widehat{\tilde{R}} = \bigoplus_{i=1}^{I} \widehat{\tilde{R}}_i = \bigoplus_{i=1}^{I} \left(\tilde{C}_{i,I-i} \otimes \left(\bigotimes_{j=I-i}^{J-1} \hat{\tilde{f}}_j - 1 \right) \right)$$

Single Accident Years

Predictor (Ultimate claims uncertainty)

Given the observations $\mathcal{D}_I$ and a parameter $K \in \mathbb{R}^+$, the uncertainty of the ultimate claims $\hat{\tilde{C}}_{i,J}$ for each accident year $i = 1, \ldots, I$ is given by

$$Unc_K\left(\hat{\tilde{C}}_{i,J} \mid \mathcal{D}_I\right) = \tfrac{1}{2} K \tilde{C}_{i,I-i}\left(l_{ult} + r_{ult}\right)$$

where $\tilde{C}_{i,I-i} = (C_{i,I-i}, 0, 0)$.

l_{ult}: left spread of product of all FCL factors up to ultimate claim
r_{ult}: right spread of product of all FCL factors up to ultimate claim

Aggregated Accident Years

Predictor (Uncertainty of the sum of ultimate claims)

Given the observations $\mathcal{D}_I$ and a parameter $K \in \mathbb{R}^+$, the uncertainty of the sum of the ultimate claims $\sum_{i=1}^{I} \hat{\tilde{C}}_{i,J}$ is given by:

$$Unc_K\left(\sum_{i=1}^{I} \hat{\tilde{C}}_{i,J} \mid \mathcal{D}_I\right) = \sum_{i=1}^{I} Unc_K\left(\hat{\tilde{C}}_{i,J} \mid \mathcal{D}_I\right)$$

Run-off Triangle

Taylor/Ashe data as in Mack(1993)

accident year i	development year j 0	1	2	3	4	5	6	7	8	9
0	357,848	1,124,788	1,735,330	2,218,270	2,745,596	3,319,994	3,466,336	3,606,286	3,833,515	3,901,463
1	352,118	1,236,139	2,170,033	3,353,322	3,799,067	4,120,063	4,647,867	4,914,039	5,339,085	
2	290,507	1,292,306	2,218,525	3,235,179	3,985,995	4,132,918	4,628,910	4,909,315		
3	310,608	1,418,858	2,195,047	3,757,447	4,029,929	4,381,982	4,588,268			
4	443,160	1,136,350	2,128,333	2,897,821	3,402,672	3,873,311				
5	396,132	1,333,217	2,180,715	2,985,752	3,691,712					
6	440,832	1,288,463	2,419,861	3,483,130						
7	359,480	1,421,128	2,864,498							
8	376,686	1,363,294								
9	344,014									

Fuzzy Chain-Ladder Factors

$\hat{\tilde{f}}_j$	development year 0	1	2	3	4	5	6	7	8
$\hat{f}_j$	3.4906	1.7473	1.4574	1.1739	1.1038	1.0863	1.0539	1.0766	1.0177
$\hat{l}_{\hat{f}_j}$	2.4906	0.7473	0.4574	0.1739	0.1038	0.0863	0.0539	0.0766	0.0177
$\hat{r}_{\hat{f}_j}$	2.4906	0.7473	0.4574	0.1739	0.1038	0.0863	0.0539	0.0766	0.0177

	development year j									
$\hat{\hat{C}}_{i,j}$	0	1	2	3	4	5	6	7	8	9
$\hat{C}_{0,j}$	357,848	1,124,788	1,735,330	2,218,270	2,745,596	3,319,994	3,466,336	3,606,286	3,833,515	3,901,463
$l_{\hat{C}_{0,j}}$	0	0	0	0	0	0	0	0	0	0
$r_{\hat{C}_{0,j}}$	0	0	0	0	0	0	0	0	0	0
$\hat{C}_{1,j}$	352,118	1,236,139	2,170,033	3,353,322	3,799,067	4,120,063	4,647,867	4,914,039	5,339,085	5,433,719
$l_{\hat{C}_{1,j}}$	0	0	0	0	0	0	0	0	0	94,634
$r_{\hat{C}_{1,j}}$	0	0	0	0	0	0	0	0	0	94,634
$\hat{C}_{2,j}$	290,507	1,292,306	2,218,525	3,235,179	3,985,995	4,132,918	4,628,910	4,909,315	5,285,148	5,378,826
$l_{\hat{C}_{2,j}}$	0	0	0	0	0	0	0	0	375,833	469,511
$r_{\hat{C}_{2,j}}$	0	0	0	0	0	0	0	0	375,833	482,834
$\hat{C}_{3,j}$	310,608	1,418,858	2,195,047	3,757,447	4,029,929	4,381,982	4,588,268	4,835,458	5,205,637	5,297,906
$l_{\hat{C}_{3,j}}$	0	0	0	0	0	0	0	247,190	617,369	709,638
$r_{\hat{C}_{3,j}}$	0	0	0	0	0	0	0	247,190	655,217	770,712
$\hat{C}_{4,j}$	443,160	1,136,350	2,128,333	2,897,821	3,402,672	3,873,311	4,207,459	4,434,133	4,773,589	4,858,200
$l_{\hat{C}_{4,j}}$	0	0	0	0	0	0	334,148	560,822	900,278	984,889
$r_{\hat{C}_{4,j}}$	0	0	0	0	0	0	334,148	596,826	1,027,662	1,148,703
$\hat{C}_{5,j}$	396,132	1,333,217	2,180,715	2,985,752	3,691,712	4,074,999	4,426,546	4,665,023	5,022,155	5,111,171
$l_{\hat{C}_{5,j}}$	0	0	0	0	0	383,287	734,834	973,311	1,330,443	1,419,459
$r_{\hat{C}_{5,j}}$	0	0	0	0	0	383,287	800,966	1,125,746	1,655,241	1,802,935
$\hat{C}_{6,j}$	440,832	1,288,463	2,419,861	3,483,130	4,088,678	4,513,179	4,902,528	5,166,649	5,562,182	5,660,771
$l_{\hat{C}_{6,j}}$	0	0	0	0	605,548	1,030,049	1,419,398	1,683,519	2,079,052	2,177,641
$r_{\hat{C}_{6,j}}$	0	0	0	0	605,548	1,155,789	1,744,557	2,196,651	2,928,515	3,130,917
$\hat{C}_{7,j}$	359,480	1,421,128	2,864,498	4,174,756	4,900,545	5,409,337	5,875,997	6,192,562	6,666,635	6,784,799
$l_{\hat{C}_{7,j}}$	0	0	0	1,310,258	2,036,047	2,544,839	3,011,499	3,328,064	3,802,137	3,920,301
$r_{\hat{C}_{7,j}}$	0	0	0	1,310,258	2,491,628	3,517,799	4,591,416	5,402,700	6,703,982	7,059,799
$\hat{C}_{8,j}$	376,686	1,363,294	2,382,128	3,471,744	4,075,313	4,498,426	4,886,502	5,149,760	5,544,000	5,642,266
$l_{\hat{C}_{8,j}}$	0	0	1,018,834	2,108,450	2,712,019	3,135,132	3,523,208	3,786,466	4,180,706	4,278,972
$r_{\hat{C}_{8,j}}$	0	0	1,018,834	3,040,506	4,701,269	6,100,587	7,541,250	8,617,067	10,330,671	10,795,153
$\hat{C}_{9,j}$	344,014	1,200,818	2,098,228	3,057,984	3,589,620	3,962,307	4,304,132	4,536,015	4,883,270	4,969,825
$l_{\hat{C}_{9,j}}$	0	856,804	1,754,214	2,713,970	3,245,606	3,618,293	3,960,118	4,192,001	4,539,256	4,625,811
$r_{\hat{C}_{9,j}}$	0	856,804	3,034,848	6,770,961	9,656,884	1,203,4794	14,453,088	16,242,272	19,076,387	19,839,189

Estimated Reserves

accident-year i	$\hat{\tilde{R}}_i = (\hat{R}_i, \hat{l}_{\hat{R}_i}, \hat{r}_{\hat{R}_i})$		
	$\hat{R}_i$	$\hat{l}_{\hat{R}_i}$	$\hat{r}_{\hat{R}_i}$
0	0.00	0.00	0.00
1	94,633.81	94,633.81	94,633.81
2	469,511.29	469,511.29	482,834.38
3	709,637.82	709,637.82	770,712.24
4	984,888.64	984,888.64	1,148,703.01
5	1,419,459.46	1,419,459.46	1,802,935.09
6	2,177,640.62	2,177,640.62	3,130,917.40
7	3,920,301.01	3,920,301.01	7,059,798.97
8	4,278,972.26	4,278,972.26	10,795,153.00
9	4,625,810.69	4,625,810.69	19,839,189.18
$\sum$	18,680,855.61	18,680,855.61	45,124,877.08

Expected Reserves

accident-year i	$E_\beta(\hat{\tilde{R}}_i)$ $\beta = 0.4$	$\beta = 0.5$	chain-ladder reserve $\hat{R}_i$
0	0.00	0.00	0.00
1	85,170.43	94,633.81	94,633.81
2	425,224.78	472,842.06	469,511.29
3	650,888.92	724,906.43	709,637.82
4	919,162.65	1,025,842.23	984,888.64
5	1,354,208.64	1,515,328.37	1,419,459.46
6	2,150,531.91	2,415,959.81	2,177,640.62
7	4,156,170.50	4,705,175.50	3,920,301.01
8	5,154,311.18	5,908,017.45	4,278,972.26
9	7,205,905.32	8,429,155.31	4,625,810.69
$\sum$	22,101,574.34	25,291,860.98	18,680,855.61

Uncertainty

accident-year i	$\text{Unc}_K(\hat{\hat{C}}_{i,J} \mid \mathcal{D}_I)$ $K = 0.5$	$K = 1$	$K = 2$
0	0.00	0.00	0.00
1	47316.91	94633.81	189267.63
2	238086.42	476172.84	952345.67
3	370087.51	740175.03	1480350.06
4	533397.91	1066795.82	2133591.65
5	805598.64	1611197.27	3222394.55
6	1327139.50	2654279.01	5308558.02
7	2745025.00	5490049.99	10980099.98
8	3768531.32	7537062.63	15074125.26
9	6116249.97	12232499.94	24464999.87
$\sum$	15951433.17	31902866.35	63805732.69

Thank you for your attention!