Combining Chain-Ladder Claims Reserving with Fuzzy Numbers

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Abstract

In this paper we extend the classical chain-ladder claims reserving method using fuzzy methods. Therefore, we derive new estimators for the claims development factors as well as new predictors for the ultimate claims. The advantage in using fuzzy numbers lies in the fact that the model uncertainty is directly included in and can be controlled by the "new" fuzzy claims development factors. We also provide an estimator for the uncertainty of the ultimate claims for single accident years and for aggregated accident years.

Keywords: Claims reserving, chain-ladder model, fuzzy numbers, ultimate claims predictor, fuzzy uncertainty

JEL Classification: C10, G22

1 Introduction

1.1 Motivation

Actuaries in insurance companies are faced with the problem that reserves for outstanding claims need to be estimated in an appropriate way. There exist a number of purely computational and stochastic models. One of the methods widely used in practice is the chain-ladder (CL) method (cf. Wüthrich and Merz [28] and Mack [18]). Even though the development factors calculated in the CL model are crisp, actuaries tend to adjust the factors afterwards due to their subjective judgement. Therefore, the developement factors are not crisp any longer but vague. We propose a model in which the CL factors

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do not need to be adjusted retrospectively and, nevertheless, are flexible and include uncertainty.

Fuzzy Set Theory (FST) as introduced in Zadeh [29] has been used for many different applications in insurance. It all started off with the work of Wit [27] in which FST is applied to underwriting. A similar approach is done in Lemaire [17].

A survey of applications can be found in Shapiro [23]. Previously stated articles can be classified as insurance or actuarial science. However, the field of claims reserving within actuarial science has not received much attention in context with fuzzy sets. Andrés Sánchez and Terceño Gómez [4] propose an application of fuzzy regression (FR) to the London chain-ladder Method by Benjamin and Eagles [7] in order to determine Incurred But Not Reported (IBNR) reserves. They make use of FR as described in Tanaka and Ishibuchi [25].

Andrés Sánchez [1] suggests a method for claims reserving which applies a FR technique by Ishibuchi and Nii [15] to a claims reserving scheme proposed by Sherman [24]. Andrés Sánchez [2] combines FR with Taylor's geometric seperation method as described in Taylor [26]. Başer and Apaydin [6] apply hybrid fuzzy least-squares regression analysis as suggested by Chang [9] to the London chain-ladder Method. A recent work by Andrés Sánchez [3] applies FR to a claims reserving method suggested by Kremer [16].

The articles dealing with claims reserving all have in common that they use FR to obtain the reserves. In this paper we enhance the classical CL method by using fuzzy numbers (FN) and fuzzy arithmetic. We obtain our results by using triangular FN (TFN) which have been introduced as a special case of L-R FN by Dubois and Prade (cf. Dubois and Prade [11] or [10]). By doing so no information especially regarding the uncertainty of a quantity is lost.

The structure of the paper is as follows. In the next section fuzzy numbers and fuzzy arithmetic is introduced. In Section 2 we present the fuzzy chain-ladder (FCL) model. In Section 3 we describe how to obtain the claims reserves. Section 4 is dealing with the model uncertainty using a measurement of uncertainty proposed by Pal and Bezdek [21]. In Section 5 an example is presented. The article ends with a conclusion.

1.2 Chain-ladder method

The CL method is one of the widely used claims reserving methods in practice due to its simplicity and nonetheless good results. It assumes that the increase of the cumulative claims from one development year to another acts on average like in the previous accident

years.
In the following we denote with $C_{i,j}$ cumulative claims made in accident year $i \in \{0, \dots, I\}$ and development year $j \in \{0, \dots, J\}$. We assume that we are at time (calendar year) $t = I$, i.e. we have the following set of given observations:

$$\mathcal{O}_I = \{C_{i,j} \mid i + j \leq I\} \tag{1.1}$$

Figure 1 demonstrates the specifications made above. The upper left part in the given development triangle is observable at time $t = I$, while the lower right part is unobservable. For simplification we only consider the case $I = J$, i.e. development triangles. Of course all formulas also hold true for development trapezoids.

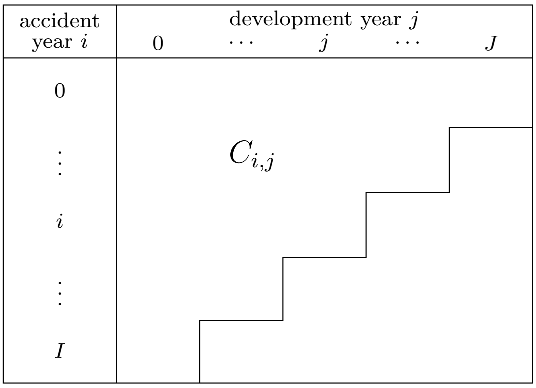

Figure 1: Development triangle at time $t = I$ with observable cumulative claims $C_{i,j}$ in the upper left part, while the lower right part is unobservable.

The CL methods's objective is to fill the development triangle, especially calculate the ultimative claims. This is done with so called claims development factors (or CL-factors):

$$\hat{f}_j = \frac{\sum_{i=0}^{I-j-1} C_{i,j+1}}{\sum_{i=0}^{I-j-1} C_{i,j}}, \quad j = 0, \dots, J-1$$

We yield the full triangle and the ultimate claims by successively multiplying the diagonal elements with the CL factors, i.e. $C_{I-i,i} \cdot f_{I-i} \cdot \ldots \cdot f_{J-1}$. Subsequently, the reserves are derived.

1.3 Fuzzy numbers and fuzzy arithmetic

In our fuzzy chain-ladder (FCL) model we describe the unobservable chain-ladder development factors introduced in Section 2 as FNs. To define these FNs we have to define fuzzy subsets in advance.

Definition 1.1 (Fuzzy subsets)**:** *A fuzzy subset* $\tilde{A}$ *of* $\mathbb{R}$ *is defined as*

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in \mathbb{R}\} \tag{1.2}$$

with membership function $\mu_{\tilde{A}}$ *given by:*

$$\mu_{\tilde{A}} : \mathbb{R} \to [0, 1] \tag{1.3}$$

Remarks 1.2:

- The fuzzy subset $\tilde{A}$ is called "normal" subset if and only if there exists an $x \in \mathbb{R}$ with $\mu_{\tilde{A}}(x) = 1$.
- The membership function $\mu_{\tilde{A}}$ describes to what extent an element $x \in \mathbb{R}$ belongs to the fuzzy set $\tilde{A}$.

Figure 2 is an example of a (triangular) membership function $\mu_{\tilde{A}}$.

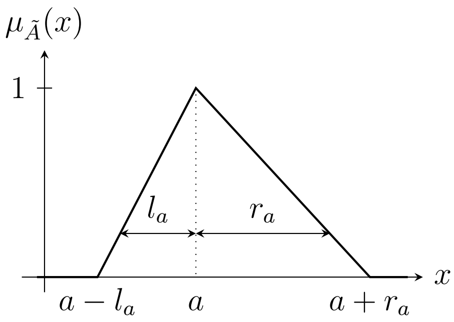

Figure 2: A TFN $\tilde{A} = (a, l_a, r_a)$.

In the following we are only working with triangular membership functions and the corresponding triangular fuzzy numbers (TFNs). The reason for using TFNs is given by the fact that they are easy to handle and can be interpreted intuitively.

Since we are only dealing with TFNs it is sufficient to define only these.

Definition 1.3 (Triangular fuzzy numbers (TFNs))**:** *A triangular fuzzy number is characterized by its membership function* $\mu_{\tilde{A}}$ *given by*

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a+l_a}{l_a} & \textit{if } a - l_a < x \leq a \\ \frac{a+r_a-x}{r_a} & \textit{if } a < x \leq a + r_a \\ 0 & \textit{otherwise} \end{cases} \tag{1.4}$$

for all $a, x \in \mathbb{R}$ *and* $l_a, r_a > 0$.

Remarks 1.4:

- Since a fuzzy number (FN) is a fuzzy subset $\tilde{A}$ over real numbers, a TFN can be written as $\tilde{A} = (a, l_a, r_a)$ with $a \in \mathbb{R}$ and $l_a, r_a > 0$. In Figure 2 an example for a TFN is given. In the following we will speak of a as the mode and l_a and r_a as the left and right spread of the TFN $\tilde{A}$.
- The case $l_a = r_a = 0$ is not covered by Definition 1.3 but it is still possible. In fact, the TFN $\tilde{A} = (a, 0, 0)$ is just a real/crisp number $a \in \mathbb{R}$.
- If $l_a = 0$ (respectively $r_a = 0$) while $r_a > 0$ ($l_a > 0$) the TFN $\tilde{A}$ is a left-sided (right-sided) degenerated TFN.
- A TFN $\tilde{A} = (a, l_a, r_a)$ is said to be non-negative if and only if $a \geq l_a$ and positive if and only if $a > l_a$.
- From a statistical point of view the membership function $\mu_{\tilde{X}}$ plays a similar role as a density function based on a random variable X.

In the following some basic fuzzy arithmetic is introduced (cf. Bansal [5]).

Definition 1.5 (Fuzzy arithmetic)**:** *Let $\tilde{A} = (a, l_a, r_a)$ and $\tilde{B} = (b, l_b, r_b)$. The (algebraic) sum, product and inverse over $\mathbb{R}$ are defined as*

$$\tilde{C} = \tilde{A} \oplus \tilde{B} = (a, l_a, r_a) \oplus (b, l_b, r_b) = (a + b, l_a + l_b, r_a + r_b) \tag{1.5}$$

$$\tilde{E} = \tilde{A} \otimes \tilde{B} = (a, l_a, r_a) \otimes (b, l_b, r_b) \approx (ab, al_b + bl_a - l_a l_b, ar_b + br_a + r_a r_b) \tag{1.6}$$

$$\tilde{F} = \tilde{A}^{-1} = 1/\tilde{A} = \left(\frac{1}{a}, \frac{r_a}{a(a + r_a)}, \frac{l_a}{a(a - l_a)} \right) \tag{1.7}$$

for all $a, b, l_a, r_a, l_b, r_b > 0$ with $a > l_a$ and $b > l_b$.

Remarks 1.6:

- Definition 1.5 only holds true if and only if the two TFNs $\tilde{A}$ and $\tilde{B}$ are positive TFNs.
- Multiplication is not a closed operation for TFNs as defined above. We use tangent approximation in the sense of Hanss [13, p. 57]. Knowing of this we will nevertheless omit the approximation in the following and write "=" instead.
- With the help of the definition of an inverse TFN the quotient $\tilde{A} \oslash \tilde{B}$ of two TFNs $\tilde{A}$ and $\tilde{B}$ is also given.

Example 1.7: Let $\tilde{A}$ and $\tilde{B}$ be two non-negative TFNs given by

$$\tilde{A} = (6, 2, 3) \qquad \text{and} \qquad \tilde{B} = (5, 4, 1).$$

The operations defined in Definition 1.5 are illustrated by the following results:

$$\tilde{C} = \tilde{A} \oplus \tilde{B} = (6, 2, 3) \oplus (5, 4, 1) = (11, 6, 4) \tag{1.8}$$

$$\tilde{E} = \tilde{A} \otimes \tilde{B} = (6, 2, 3) \otimes (5, 4, 1) = (30, 26, 24) \tag{1.9}$$

$$\tilde{F} = \tilde{A} \oslash \tilde{B} = (6, 2, 3) \oslash (5, 4, 1) = (1.2, 0.\overline{6}, 0.5\overline{3}) \tag{1.10}$$

The TFNs $\tilde{A}$ and $\tilde{B}$ as well as the results given in equations (1.8)-(1.10) are displayed in the Figures 3, 4 and 5.

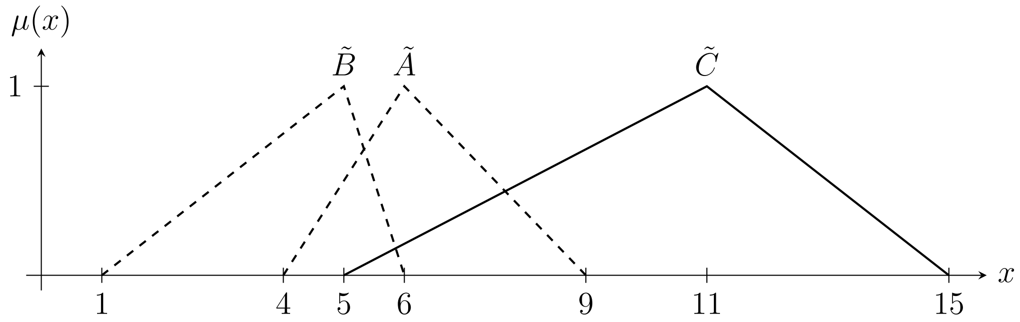

Figure 3: Fuzzy-sum of the TFNs $\tilde{A} = (6, 2, 3)$ and $\tilde{B} = (5, 4, 1)$.

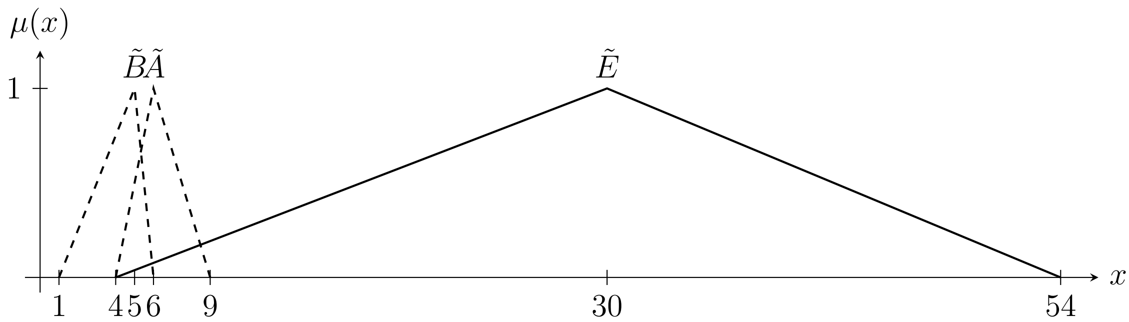

Figure 4: Fuzzy-product of the TFNs $\tilde{A} = (6, 2, 3)$ and $\tilde{B} = (5, 4, 1)$.

Not to overload notation we denote the fuzzy-sum "$\tilde{A} \oplus \tilde{B}$" by "$\tilde{A} + \tilde{B}$", the fuzzy-product "$\tilde{A} \otimes \tilde{B}$" by "$\tilde{A}\tilde{B}$" and the fuzzy-quotient "$\tilde{A} \oslash \tilde{B}$" by "$\tilde{A}/\tilde{B}$", respectively "$\frac{\tilde{A}}{\tilde{B}}$". In the same way we denote the product of n FNs $\tilde{A}_1, \ldots, \tilde{A}_n$ by $\prod_{i=1}^{n} \tilde{A}$.

In the following a definition for the expected value of TFNs is given (cf. Campos Ibáñez and González Muñoz [8] or Andrés Sánchez [2]). This definition is also considered in the context of Heilpern [14].

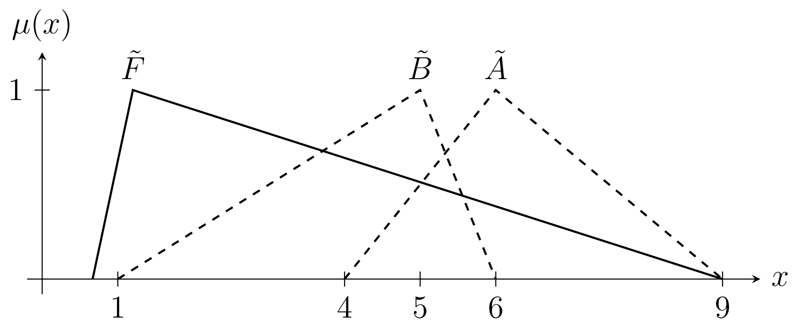

Figure 5: Fuzzy-quotient of the TFNs $\tilde{A} = (6, 2, 3)$ and $\tilde{B} = (5, 4, 1)$.

Definition 1.8 (Expected value of a TFN)**:** *Let $\tilde{A} = (a, l_a, r_a)$ and $\tilde{B}$ be TFNs and $0 \leq \beta \leq 1$. The expected value of the TFN $\tilde{A}$ (denoted with $E_\beta(\tilde{A})$) is given by:*

$$E_\beta(\tilde{A}) = a - \frac{1-\beta}{2} l_a + \frac{\beta}{2} r_a \tag{1.11}$$

The notation $E_\beta(\tilde{A} \mid \tilde{B})$ means that $\tilde{B}$ shall be considered as $\tilde{B} = (B, 0, 0)$, i.e. a crisp number.

Remarks 1.9:

- The parameter β is called "decision-maker risk parameter".
- If $\tilde{A}$ is a crisp number, i.e. $\tilde{A} = (a, 0, 0)$ the expected value is given by $E_\beta(\tilde{A}) = a$. This is equivalent to the stochastic version of the expected value and is independent from the choice of β.
- A choice of $\beta \geq 0.5$ leads to a risk-averse manner of reserving.

The definition of the uncertainty of a TFN is more complicated than the definition of an expected value of a TFN (cf. Definition 1.8). We use a measurement of uncertainty proposed by Pal and Bezdek [21].

Definition 1.10 (Uncertainty of a TFN)**:** *Let $\tilde{A} = (a, l_a, r_a)$ and $\tilde{B}$ be TFNs and $K \in \mathbb{R}^+$. The uncertainty of a TFN $\tilde{A}$ (denoted with $\mathrm{Unc}_K(\tilde{A})$) is given by:*

$$\mathrm{Unc}_K(\tilde{A}) = \tfrac{1}{2} K (l_a + r_a) \tag{1.12}$$

The notation $\mathrm{Unc}_K(\tilde{A}|\tilde{B})$ means that $\tilde{B}$ shall be interpreted as a crisp number, i.e. $\tilde{B} = (B, 0, 0)$.

Remarks 1.11:

- The uncertainty of a TFN $\tilde{A} = (a, l_a, r_a)$ is, as expected, independent of the mode a and depends only on the support of $\tilde{A}$.
- The uncertainty of a TFN $\tilde{A}$ can be interpreted as the (weighted) area between the x-axis and the membership function $\mu_{\tilde{A}}$.
- The definition of the uncertainty is motivated by Example 3 in Pal and Bezdek [21] applied to the TFNs given in Definition 1.3.

Example 1.12 (see Example 1.7)**:** Let $\tilde{A}$ and $\tilde{B}$ be two non-negative TFNs given by

$$\tilde{A} = (6, 2, 3) \qquad \text{and} \qquad \tilde{B} = (5, 4, 1).$$

The expected values of $\tilde{A}$ and $\tilde{B}$ and their uncertainties are given by ($\beta = 0.5$ and $K = 1$):

$$E_{0.5}(\tilde{A}) = 6.25 \qquad\qquad E_{0.5}(\tilde{B}) = 4.25$$
$$\text{Unc}_1(\tilde{A}) = 2.5 \qquad\qquad \text{Unc}_1(\tilde{B}) = 2.5$$

2 The fuzzy chain-ladder (FCL) model

Our goal in this paper is to develop an estimator for the ultimate claims $C_{i,J}$ for $i = 1, \ldots, I$ using classic CL methods combined with fuzzy numbers. We use the notations introduced in section 1.2. We also derive an estimator for the uncertainty of the predicted ultimate claims based on our Model Assumptions 2.1.

Model Assumptions 2.1 (Fuzzy chain-ladder (FCL) model)**:** There exist TFNs $\tilde{f}_j$ ($j = 0, \ldots, J-1$) so that the cumulative claims can be written as

$$\tilde{C}_{i,j+1} = \tilde{f}_j \tilde{C}_{i,j} \tag{2.1}$$

for all $i = 0, \ldots, I$ and $j = 0, \ldots, J-1$.

Remarks 2.2:

- Since every real number $a \in \mathbb{R}$ can also be denoted as TFN $\tilde{A} = (a, 0, 0)$ we consider the cumulative claims as TFNs. If $\tilde{C}_{i,j}$ is observable, i.e. if $i + j \leq I$ holds, the cumulative claims can be written as $\tilde{C}_{i,j} = (C_{i,j}, 0, 0)$.
- Equation (2.1) compared with the CL model given for example in Merz and Wüthrich [19], Gisler and Wüthrich [12] or in Peters, Wüthrich, and Shevchenko [22] seems

to have no “variance-term” σ_j as denoted in Merz and Wüthrich [19]. The reason for this is that in the FCL model the “variance”, respectively the uncertainty, is completely contained in the TFNs $\tilde{f}_j$ $(j = 0, \ldots, J-1)$. Moreover, the FCL model does not model stochastic randomness but uncertainty.

With Model Assumptions 2.1 we achieve the following result.

Lemma 2.3: *For all* $0 \leq i \leq I$ *and* $0 \leq j \leq J-1$ *let* $\tilde{C}_{i,j}$ *and* $\tilde{f}_j = (f_j, l_{f_j}, r_{f_j})$ *be non-negative fuzzy numbers. Then, we have*

$$E_\beta(\tilde{C}_{i,j+1} \mid \tilde{C}_{i,j}) = C_{i,j}\left(f_j - \frac{1-\beta}{2} l_{f_j} + \frac{\beta}{2} r_{f_j}\right) \tag{2.2}$$

and

$$\mathrm{Unc}_K(\tilde{C}_{i,j+1} \mid \tilde{C}_{i,j}) = \tfrac{1}{2} K C_{i,j}(l_{f_j} + r_{f_j}). \tag{2.3}$$

Proof. Using

$$E_\beta(\tilde{C}_{i,j+1} \mid \tilde{C}_{i,j}) = E_\beta(\tilde{f}_j \tilde{C}_{i,j} | \tilde{C}_{i,j}) = \tilde{C}_{i,j}\, E_\beta(\tilde{f}_j) \tag{2.4}$$

and

$$\mathrm{Unc}_K(\tilde{C}_{i,j+1} \mid \tilde{C}_{i,j}) = \mathrm{Unc}_K(\tilde{f}_j \tilde{C}_{i,j} \mid \tilde{C}_{i,j}) = \mathrm{Unc}_K((C_{i,j} f_j, C_{i,j} l_{f_j}, C_{i,j} r_{f_j})) \tag{2.5}$$

together with Definitions 1.5, 1.8 and 1.10 completes the proof. □

Remarks 2.4:

- In the classical CL method (cf. Wüthrich and Merz [28, p. 37]) the following conditional expectation and conditional variance are given:

$$E(C_{i,j+1} \mid C_{i,j}) = f_j C_{i,j} \tag{2.6}$$
$$\mathrm{Var}(C_{i,j+1} \mid C_{i,j}) = \sigma_j^2 C_{i,j} \tag{2.7}$$

  When choosing $\beta = 0.5$ and $l_{f_j} = r_{f_j}$ equation (2.2) results in equation (2.6). In equation (2.3) $l_{f_j} + r_{f_j}$ play the role of σ_j^2 – the measure of uncertainty, respectively the variance.

In the following we propose an estimator for the TFNs $\tilde{f}_j$ $(j = 0, \ldots, J-1)$. With these estimators, always denoted with a hat, we are able to predict the ultimate claims $\tilde{C}_{i,J}$

$(i = 1, \dots, I)$ by

$$\hat{\tilde{C}}_{i,J} = C_{i,I-i} \prod_{j=I-i}^{J-1} \hat{\tilde{f}}_j. \tag{2.8}$$

The cumulative claims $C_{i,I-i}$ $(i = 1, \dots, I)$ are observable, i.e. these claims are crisp numbers.

Estimator 2.5 (FCL estimator for the TFNs $\tilde{f}_j$)**:** *The TFNs $\tilde{f}_j = (f_j, l_{f_j}, r_{f_j})$ $(j = 0, \dots, J-1)$ introduced in Model Assumptions 2.1 are estimated by $\hat{\tilde{f}}_j = (\hat{f}_j, \hat{l}_{\hat{f}_j}, \hat{r}_{\hat{f}_j})$ with*

$$\hat{f}_j = \frac{\sum_{i=0}^{I-j-1} C_{i,j+1}}{\sum_{i=0}^{I-j-1} C_{i,j}} \tag{2.9}$$

$$\hat{l}_{\hat{f}_j} = \hat{r}_{\hat{f}_j} = \frac{\sum_{i=0}^{I-j-1} X_{i,j+1}}{\sum_{i=0}^{I-j-1} C_{i,j}} \tag{2.10}$$

where

$$X_{i,j+1} = C_{i,j+1} - C_{i,j} \tag{2.11}$$

for $i = 0, \dots, I$ and $j = 0, \dots, J-1$.

Remarks 2.6:

- Since we are choosing $\hat{l}_{f_j} = \hat{r}_{f_j}$ for all $j = 0, \dots, J-1$ we get symmetric TFNs – so called STFNs.
- The expected values $E_{0.5}(\hat{\tilde{f}}_j)$ for $j = 0, \dots, J-1$ are equal to the estimators of the CL development factors $\hat{f}_j$ (cf. Wüthrich and Merz [28, p. 38]) since the estimators $\hat{\tilde{f}}_j$ are STFNs and the "risk-parameter" is chosen as $\beta = 0.5$.
- The fact that we are dealing with cumulative claims results in a natural left-border for the development factors $\tilde{f}_j$ for $j = 0, \dots, J-1$. A lowerd bound is given by 1 and this is included in Estimator 2.5, i.e.:

$$\hat{f}_j - \hat{l}_{f_j} = \frac{\sum_{i=0}^{I-j-1} C_{i,j+1}}{\sum_{i=0}^{I-j-1} C_{i,j}} - \frac{\sum_{i=0}^{I-j-1} X_{i,j+1}}{\sum_{i=0}^{I-j-1} C_{i,j}} = \frac{\sum_{i=0}^{I-j-1}(C_{i,j+1} - C_{i,j+1} + C_{i,j})}{\sum_{i=0}^{I-j-1} C_{i,j}} = 1 \tag{2.12}$$

- The values $X_{i,j+1}$ $(i = 0, \dots, I$ and $j = 0, \dots, J-1)$ are observable incremental claims.

3 Claims reserves

Equation (2.8) leads to a predictor for the cumulative claims for accident year $i = 1, \dots, I$. This predictor is used to introduce a formula for an estimator of the claims reserve for single accident years. The claims reserve $\tilde{R}_i$ for a given accident year $i = 1, \dots, I$ is given by

$$\tilde{R}_i = \tilde{C}_{i,J} - \tilde{C}_{i,I-i} \tag{3.1}$$

where $\tilde{C}_{i,I-i}$ is observable, i.e. $\tilde{C}_{i,I-i} = (C_{i,I-i}, 0, 0)$. Since, at time $t = I$, the ultimate claims $\tilde{C}_{i,J}$ are unknown an estimator for the claims reserves is given by

$$\hat{\tilde{R}}_i = \hat{\tilde{C}}_{i,J} - \tilde{C}_{i,I-i} = \tilde{C}_{i,I-i} \left(\prod_{j=I-i}^{J-1} \hat{\tilde{f}}_j - 1 \right) \tag{3.2}$$

for $i = 1, \dots, I$ where $\tilde{C}_{i,I-i} = (C_{i,I-i}, 0, 0)$.

The aggregated claims reserve is given by

$$\tilde{R} = \sum_{i=1}^{I} \tilde{R}_i = \sum_{i=1}^{I} \left(\tilde{C}_{i,J} - \tilde{C}_{i,I-i} \right) \tag{3.3}$$

and an estimator for the aggregated claims reserve is:

$$\hat{\tilde{R}} = \sum_{i=1}^{I} \hat{\tilde{R}}_i = \sum_{i=1}^{I} \left(\tilde{C}_{i,I-i} \left(\prod_{j=I-i}^{J-1} \hat{\tilde{f}}_j - 1 \right) \right) \tag{3.4}$$

4 Estimation uncertainty

4.1 Single accident years

In the classical CL model the calculation of the estimation uncertainty is done with the conditional mean square error of prediction (MSEP). The conditional MSEP in the classical CL model is given by (cf. Wüthrich and Merz [28, p. 41])

$$\text{MSEP}(\widehat{C}_{i,J} \mid \mathcal{O}_I) = E\left(\left(\widehat{C}_{i,J} - C_{i,J} \right)^2 \,\middle|\, \mathcal{O}_I \right) \tag{4.1}$$

for $i = 1, \dots, I$.

In classical CL models the calculation of the conditional MSEP could be done with (different) conditional and unconditional resampling methods (cf. Wüthrich and Merz [28,

p. 45]). These methods are used due to the fact that the ultimate claims $C_{i,J}$ $(i = 1, \ldots, I)$ are unobservable. The referred resampling methods are possible since the CL model could be written as a time series model with a stochastic term. Since in our FCL model there is no stochastic term the calculation of the estimation uncertainty is done in a different way.

We calculate the estimation uncertainty using Definition 1.5. Since, in our model, the last observation in accident year $i \in \{1, \ldots, I\}$ is multiplied with at least one or more TFNs $\tilde{f}_j$ $(j = 0, \ldots, J-1)$, the uncertainty for increasing accident years $i \in \{1, \ldots, I\}$ is increasing as well.

Prior to deriving an estimator for the uncertainty we need to calculate the product of different FCL factors $\hat{\tilde{f}}_j$ $(j = 0, \ldots, J-1)$.

Lemma 4.1 (Fuzzy product of different chain-ladder factors)**:** *Given the estimators* $\hat{\tilde{f}}_j$ $(j = 0, \ldots, J-1)$ *of the fuzzy chain-ladder factors* $\tilde{f}_j$ $(j = 0, \ldots, J-1)$ *it follows that*

$$\prod_{j=I-i}^{J-1} \hat{\tilde{f}}_j = \left(\hat{F}_{I-i}^{J-1}, \hat{l}_{\hat{F}_{I-i}^{J-1}}, \hat{r}_{\hat{F}_{I-i}^{J-1}} \right) \tag{4.2}$$

holds with

$$\hat{F}_{I-i}^{J-1} := \prod_{j=I-i}^{J-1} \hat{f}_j \tag{4.3}$$

$$\hat{l}_{\hat{F}_{I-i}^{J-1}} := \hat{F}_{I-i}^{J-1} - 1 \tag{4.4}$$

$$\hat{r}_{\hat{F}_{I-i}^{J-1}} := \prod_{j=I-i}^{J-1} (2\hat{f}_j - 1) - \hat{F}_{I-i}^{J-1} \tag{4.5}$$

for $i = 1, \ldots, I$*.*

Proof. See Appendix. □

Remark 4.2: An iterative representation of the right spread $r_{\hat{F}_{I-i}^{J-1}}$ is given by $r_{\hat{F}_{I-i}^{J-1}} := (\hat{f}_{I-i} - 1) \prod_{j=I-i+1}^{J-1} \hat{f}_j + \hat{f}_{I-i} r_{\hat{F}_{I-i+1}^{J-1}} + r_{\hat{F}_{I-i+1}^{J-1}} (\hat{f}_{I-i} - 1)$.

With Lemma 4.1 the ultimate claim uncertainty for a single accident year is given by the following estimator.

Estimator 4.3 (Ultimate claims uncertainty)**:** *Given the observations* $\mathcal{O}_I$ *and a parameter* $K \in \mathbb{R}^+$*, the uncertainty of the ultimate claims* $\hat{\tilde{C}}_{i,J}$ *for each accident year* $i = 1, \ldots, I$ *is given by*

$$\mathrm{Unc}_K \left(\hat{\tilde{C}}_{i,J} \mid \mathcal{O}_I \right) = \tfrac{1}{2} K \tilde{C}_{i,I-i} \left(l_{\hat{F}_{I-i}^{J-1}} + r_{\hat{F}_{I-i}^{J-1}} \right) \tag{4.6}$$

where $\tilde{C}_{i,I-i} = (C_{i,I-i}, 0, 0)$.

Proof. The estimator for the ultimate claim $\tilde{C}_{i,J}$ $(i = 1, \dots, I)$ given the observation $\mathcal{O}_I$ can be displayed as:

$$\hat{\tilde{C}}_{i,J} = \tilde{C}_{i,I-i} \prod_{j=I-i}^{J-1} \hat{\tilde{f}}_j \tag{4.7}$$

Since, at time $t = I$, cumulative claims $\tilde{C}_{i,I-i}$ $(i = 1, \dots, I)$ are observable, respectively crisp numbers, the uncertainty of the ultimate claims is given by:

$$\begin{aligned}
\mathrm{Unc}_K\left(\hat{\tilde{C}}_{i,J} \mid \mathcal{O}_I\right) &= \mathrm{Unc}_K\left(\tilde{C}_{i,I-i} \prod_{j=I-i}^{J-1} \hat{\tilde{f}}_j \mid \mathcal{O}_I\right) \\
&= \mathrm{Unc}_K\left((C_{i,I-i}, 0, 0) \otimes \left(\hat{F}_{I-i}^{J-1}, \hat{l}_{\hat{F}_{I-i}^{J-1}}, \hat{r}_{\hat{F}_{I-i}^{J-1}}\right)\right) \\
&= \mathrm{Unc}_K\left(\left(C_{i,I-i}\hat{F}_{I-i}^{J-1}, C_{i,I-i}\hat{l}_{\hat{F}_{I-i}^{J-1}}, C_{i,I-i}\hat{r}_{\hat{F}_{I-i}^{J-1}}\right)\right) \\
&= \tfrac{1}{2} K C_{i,I-i}\left(\hat{l}_{\hat{F}_{I-i}^{J-1}} + \hat{r}_{\hat{F}_{I-i}^{J-1}}\right)
\end{aligned} \tag{4.8}$$

□

4.2 Aggregated accident years

The uncertainty of the cumulative ultimate claims $\sum_{i=1}^{I} \hat{\tilde{C}}_{i,J}$ given the observations $\mathcal{O}_I$ is not as difficult as in the classical CL model (cf. Wüthrich and Merz [28, pp. 55 sqq.]). Since we are not dealing with random variables the uncertainty for aggregated accident years can be calculated equal to Section 4.1.

Before calculating the uncertainty of aggregated accident years we start with the calculation of the sum of the ultimate claims given the observations $\mathcal{O}_I$.

Lemma 4.4: *Given the observations $\mathcal{O}_I$, i.e. we consider all observations as crisp numbers, the sum of the predicted ultimate claims is given by:*

$$\left(\sum_{i=1}^{I} \hat{\tilde{C}}_{i,J} \mid \mathcal{O}_I\right) = \left(\sum_{i=1}^{I} C_{i,I-i}\hat{F}_{I-i}^{J-1}, \sum_{i=1}^{I} C_{i,I-i}\hat{l}_{\hat{F}_{I-i}^{J-1}}, \sum_{i=1}^{I} C_{i,I-i}\hat{r}_{\hat{F}_{I-i}^{J-1}}\right) \tag{4.9}$$

Proof. The sum of the predicted ultimate claims given the observations $\mathcal{O}_I$ can be displayed as:

$$\left(\sum_{i=1}^{I} \hat{\tilde{C}}_{i,J} \mid \mathcal{O}_I\right) = \left(\sum_{i=1}^{I} \left[\tilde{C}_{i,I-i} \prod_{j=I-i}^{J-1} \hat{\tilde{f}}_j\right] \mid \mathcal{O}_I\right)$$
$$= \sum_{i=1}^{I} (C_{i,I-i}, 0, 0) \otimes \left(\hat{F}_{I-i}^{J-1}, \hat{l}_{\hat{F}_{I-i}^{J-1}}, \hat{r}_{\hat{F}_{I-i}^{J-1}}\right)$$
$$= \sum_{i=1}^{I} \left(C_{i,I-i}\hat{F}_{I-i}^{J-1}, C_{i,I-i}\hat{l}_{\hat{F}_{I-i}^{J-1}}, C_{i,I-i}\hat{r}_{\hat{F}_{I-i}^{J-1}}\right)$$
$$= \left(\sum_{i=1}^{I} C_{i,I-i}\hat{F}_{I-i}^{J-1}, \sum_{i=1}^{I} C_{i,I-i}\hat{l}_{\hat{F}_{I-i}^{J-1}}, \sum_{i=1}^{I} C_{i,I-i}\hat{r}_{\hat{F}_{I-i}^{J-1}}\right) \tag{4.10}$$

□

With Lemma 4.4 the uncertainty of the sum of the ultimate claims can be predicted.

Estimator 4.5 (Uncertainty of the sum of the ultimate claims)**:** *Given the observations $\mathcal{O}_I$ and a parameter $K \in \mathbb{R}^+$, the uncertainty of the sum of the ultimate claims $\sum_{i=1}^{I} \hat{\tilde{C}}_{i,J}$ is given by:*

$$\mathrm{Unc}_K\left(\sum_{i=1}^{I} \hat{\tilde{C}}_{i,J} \mid \mathcal{O}_I\right) = \sum_{i=1}^{I} \mathrm{Unc}_K\left(\hat{\tilde{C}}_{i,J} \mid \mathcal{O}_I\right) \tag{4.11}$$

Proof. Follows directly from Definition 1.10, Estimator 4.3 and Lemma 4.4. □

5 Example

For our example we use a run-off-triangle given in Murphy [20]. The triangle is given in Table 1.

accident-year i	development year j 0	1	2	3	4	5	6	7	8	9
0	357848	1124788	1735330	2218270	2745596	3319994	3466336	3606286	3833515	3901463
1	352118	1236139	2170033	3353322	3799067	4120063	4647867	4914039	5339085	
2	290507	1292306	2218525	3235179	3985995	4132918	4628910	4909315		
3	310608	1418858	2195047	3757447	4029929	4381982	4588268			
4	443160	1136350	2128333	2897821	3402672	3873311				
5	396132	1333217	2180715	2985752	3691712					
6	440832	1288463	2419861	3483130						
7	359480	1421128	2864498							
8	376686	1363294								
9	344014									

Table 1: Observed cumulative claims $C_{i,j}$.

The computed fuzzy chain-ladder factors $\hat{\tilde{f}}_j$ $(j = 0, \ldots, J-1)$ are given in Table 2.

$\hat{\tilde{f}}_j$	development year								
	0	1	2	3	4	5	6	7	8
$\hat{f}_j$	3.4906	1.7473	1.4574	1.1739	1.1038	1.0863	1.0539	1.0766	1.0177
$\hat{l}_{\hat{f}_j}$	2.4906	0.7473	0.4574	0.1739	0.1038	0.0863	0.0539	0.0766	0.0177
$\hat{r}_{\hat{f}_j}$	2.4906	0.7473	0.4574	0.1739	0.1038	0.0863	0.0539	0.0766	0.0177

Table 2: Fuzzy chain-ladder factors $\hat{\tilde{f}}_j$ for $j = 0, \ldots, J-1$.

With the calculated fuzzy chain-ladder factors we are able to predict the unobservable triangle, i.e. we are able to fill the lower right part of the of the run-off-triangle given in Table 1. Due to the fact that each predicted "future" claim $\tilde{C}_{i,j}$ ($i = 0, \ldots, I$ and $j = 0, \ldots, J$ with $i + j > I$) is a fuzzy number, we write down these results for each accident year $i = 0, \ldots, I$ in three lines (cf. Table 3).

In Table 4 the reserves $\hat{\tilde{R}}_i$ for $i = 0, \ldots, I$ are given. The FCL method allows the reserve amounting to zero. But the value of the membership function referring to this event is small since the slope of the membership function in the interval between the left spread and the mode is very small.

The expected reserves $E_\beta(\hat{\tilde{R}}_i)$ ($i = 0, \ldots, I$) for different choices of the "decision-maker risk parameter" β are given in Table 5 and compared with the reserves $\hat{R}_i$ ($i = 0, \ldots, I$) of the classical chain-ladder framework. It turns out that a choice of $\beta \geq 0.5$ leads to risk-averse estimation of the reserves. Moreover, the spreads of the fuzzy numbers derived by Estimator 2.5 are calculated carefully.

In Table 6 the estimation uncertainty $\text{Unc}_K(\hat{\tilde{C}}_{i,J} \mid \mathcal{O}_I)$ for different choices of the parameter $K \in \mathbb{R}^+$ for each accident year $i = 0, \ldots, I$ are given.

6 Conclusion

We have shown that in our fuzzy chain-ladder model the uncertainty in the estimation of the ultimate claims could be determined in advance by determining the range of the fuzzy chain-ladder factors $\tilde{f}_j$ ($j = 0, \ldots, J-1$). In our derivation the left border of the fuzzy chain-ladder factors $\tilde{f}_j$ ($j = 0, \ldots, J-1$) was set equal to 1, but this is not mandatory.

We have also seen that the results for the reserve-estimates, for specific choices of the "decision-maker risk parameter" β are very similar to the classical chain-ladder framework given for example in Wüthrich and Merz [28].

	development year j									
$\hat{\tilde{C}}_{i,j}$	0	1	2	3	4	5	6	7	8	9
$\hat{C}_{0,j}$	357848	1124788	1735330	2218270	2745596	3319994	3466336	3606286	3833515	3901463
$\hat{l}_{\hat{C}_{0,j}}$	0	0	0	0	0	0	0	0	0	0
$\hat{r}_{\hat{C}_{0,j}}$	0	0	0	0	0	0	0	0	0	0
$\hat{C}_{1,j}$	352118	1236139	2170033	3353322	3799067	4120063	4647867	4914039	5339085	5433719
$\hat{l}_{\hat{C}_{1,j}}$	0	0	0	0	0	0	0	0	0	94634
$\hat{r}_{\hat{C}_{1,j}}$	0	0	0	0	0	0	0	0	0	94634
$\hat{C}_{2,j}$	290507	1292306	2218525	3235179	3985995	4132918	4628910	4909315	5285148	5378826
$\hat{l}_{\hat{C}_{2,j}}$	0	0	0	0	0	0	0	0	375833	469511
$\hat{r}_{\hat{C}_{2,j}}$	0	0	0	0	0	0	0	0	375833	482834
$\hat{C}_{3,j}$	310608	1418858	2195047	3757447	4029929	4381982	4588268	4835458	5205637	5297906
$\hat{l}_{\hat{C}_{3,j}}$	0	0	0	0	0	0	0	247190	617369	709638
$\hat{r}_{\hat{C}_{3,j}}$	0	0	0	0	0	0	0	247190	655217	770712
$\hat{C}_{4,j}$	443160	1136350	2128333	2897821	3402672	3873311	4207459	4434133	4773589	4858200
$\hat{l}_{\hat{C}_{4,j}}$	0	0	0	0	0	0	334148	560822	900278	984889
$\hat{r}_{\hat{C}_{4,j}}$	0	0	0	0	0	0	334148	596826	1027662	1148703
$\hat{C}_{5,j}$	396132	1333217	2180715	2985752	3691712	4074999	4426546	4665023	5022155	5111171
$\hat{l}_{\hat{C}_{5,j}}$	0	0	0	0	0	383287	734834	973311	1330443	1419459
$\hat{r}_{\hat{C}_{5,j}}$	0	0	0	0	0	383287	800966	1125746	1655241	1802935
$\hat{C}_{6,j}$	440832	1288463	2419861	3483130	4088678	4513179	4902528	5166649	5562182	5660771
$\hat{l}_{\hat{C}_{6,j}}$	0	0	0	0	605548	1030049	1419398	1683519	2079052	2177641
$\hat{r}_{\hat{C}_{6,j}}$	0	0	0	0	605548	1155789	1744557	2196651	2928515	3130917
$\hat{C}_{7,j}$	359480	1421128	2864498	4174756	4900545	5409337	5875997	6192562	6666635	6784799
$\hat{l}_{\hat{C}_{7,j}}$	0	0	0	1310258	2036047	2544839	3011499	3328064	3802137	3920301
$\hat{r}_{\hat{C}_{7,j}}$	0	0	0	1310258	2491628	3517799	4591416	5402700	6703982	7059799
$\hat{C}_{8,j}$	376686	1363294	2382128	3471744	4075313	4498426	4886502	5149760	5544000	5642266
$\hat{l}_{\hat{C}_{8,j}}$	0	0	1018834	2108450	2712019	3135132	3523208	3786466	4180706	4278972
$\hat{r}_{\hat{C}_{8,j}}$	0	0	1018834	3040506	4701269	6100587	7541250	8617067	10330671	10795153
$\hat{C}_{9,j}$	344014	1200818	2098228	3057984	3589620	3962307	4304132	4536015	4883270	4969825
$\hat{l}_{\hat{C}_{9,j}}$	0	856804	1754214	2713970	3245606	3618293	3960118	4192001	4539256	4625811
$\hat{r}_{\hat{C}_{9,j}}$	0	856804	3034848	6770961	9656884	12034794	14453088	16242272	19076387	19839189

Table 3: Filled run-off-triangle with observed cumulative claims $C_{i,j}$ $(i + j \leq I)$ and predicted cumulative claims $\hat{\tilde{C}}_{i,j}$ $(i + j > I)$.

accident-	$\hat{\tilde{R}}_i = (\hat{R}_i, \hat{l}_{\hat{R}_i}, \hat{r}_{\hat{R}_i})$		
year i	$\hat{R}_i$	$\hat{l}_{\hat{R}_i}$	$\hat{r}_{\hat{R}_i}$
0	0.00	0.00	0.00
1	94633.81	94633.81	94633.81
2	469511.29	469511.29	482834.38
3	709637.82	709637.82	770712.24
4	984888.64	984888.64	1148703.01
5	1419459.46	1419459.46	1802935.09
6	2177640.62	2177640.62	3130917.40
7	3920301.01	3920301.01	7059798.97
8	4278972.26	4278972.26	10795153.00
9	4625810.69	4625810.69	19839189.18
$\sum$	18680855.61	18680855.61	45124877.08

Table 4: Estimated reserves $\hat{\tilde{R}}_i$ for $i = 0, \ldots, I$.

accident-year i	$\beta = 0.1$	$\beta = 0.25$	$E_\beta(\hat{\hat{R}}_i)$ $\beta = 0.5$	$\beta = 0.75$	$\beta = 0.9$	chain-ladder reserve $\hat{R}_i$
0	0.00	0.00	0.00	0.00	0.00	0.00
1	56780.29	70975.36	94633.81	118292.27	132487.34	94633.81
2	282372.93	353798.85	472842.06	591885.27	663311.20	469511.29
3	428836.41	539862.67	724906.43	909950.18	1020976.44	709637.82
4	599123.90	759143.28	1025842.23	1292541.19	1452560.56	984888.64
5	870849.46	1112529.05	1515328.37	1918127.68	2159807.27	1419459.46
6	1354248.21	1752390.06	2415959.81	3079529.57	3477671.42	2177640.62
7	2509155.51	3332663.00	4705175.50	6077688.00	6901195.50	3920301.01
8	2893192.39	4023751.79	5908017.45	7792283.11	8922842.50	4278972.26
9	3536155.34	5371030.33	8429155.31	11487280.30	13322155.29	4625810.69
$\sum$	12530714.44	17316144.39	25291860.98	33267577.57	38053007.52	18680855.61

Table 5: Expected values of the reserves $\hat{\hat{R}}_i$ $(i = 0, \ldots, I)$ for different choices of the "decision-maker risk parameter" β and corresponding chain-ladder reserves $\hat{R}_i$ $(i = 0, \ldots, I)$.

accident-year i	$K = 0.5$	$K = 1$	$\mathrm{Unc}_K(\hat{\hat{C}}_{i,J} \mid \mathcal{O}_I)$ $K = 2$	$K = 5$	$K = 10$
0	0.00	0.00	0.00	0.00	0.00
1	47316.91	94633.81	189267.63	473169.07	946338.15
2	238086.42	476172.84	952345.67	2380864.18	4761728.35
3	370087.51	740175.03	1480350.06	3700875.15	7401750.29
4	533397.91	1066795.82	2133591.65	5333979.12	10667958.25
5	805598.64	1611197.27	3222394.55	8055986.36	16111972.73
6	1327139.50	2654279.01	5308558.02	13271395.05	26542790.09
7	2745025.00	5490049.99	10980099.98	27450249.96	54900499.92
8	3768531.32	7537062.63	15074125.26	37685313.16	75370626.32
9	6116249.97	12232499.94	24464999.87	61162499.68	122324999.35
$\sum$	15951433.17	31902866.35	63805732.69	159514331.73	319028663.46

Table 6: Estimation uncertainty $\mathrm{Unc}_K(\hat{\hat{C}}_{i,J} \mid \mathcal{O}_I)$ for each accident year $i = 0, \ldots, I$ for different choices of $K \in \mathbb{R}^+$.

A Proof of Lemma 4.1

Let $\tilde{A} = (a, l_a, r_a)$ and $\tilde{B} = (b, l_b, r_b)$ be two fuzzy numbers. Another representation $\tilde{A}'$ for $\tilde{A}$ is given if the entries in $\tilde{A}$ do not denote the mode, left and right spread but the position on the x-axis, i.e.

$$\tilde{A}' := (a - l_a, a, a + r_a)' := (d, e, f)'.$$

An illustration is given in Figure 6. In the same way we denote $\tilde{B}$ as $\tilde{B}' = (b - l_b, b, b + r_b)' := (g, h, i)'$. The equivalent notation is always denoted with an apostrophe. With Definition 1.5

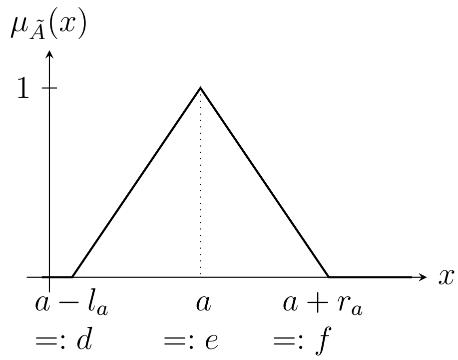

Figure 6: Equivalent notation for a TFN $\tilde{A} = (a, l_a, r_a)$.

we yield in the equivalent representation

$$\begin{aligned}\tilde{A}' \otimes \tilde{B}' &= (ab - (al_b + bl_a - l_a l_b), ab, ab + (ar_b br_a + r_a r_b))' \\ &= (gd, eh, if)'.\end{aligned}$$

Every estimator $\hat{\tilde{f}}_j$ $(j = 0, \ldots, J-1)$ can be represented as (cf. Estimator 2.5):

$$\hat{\tilde{f}}_j = (\hat{f}_j, \hat{f}_j - 1, \hat{f}_j - 1) \equiv (1, \hat{f}_j, 2\hat{f}_j - 1)' \tag{A.1}$$

Therefore we get:

$$\begin{aligned}\prod_{j=I-i}^{J-1} \hat{\tilde{f}}_j &\equiv \prod_{j=I-i}^{J-1} \left(1, \hat{f}_j, 2\hat{f}_j - 1\right)' \\ &= \left(1, \prod_{j=I-i}^{J-1} \hat{f}_j, \prod_{j=I-i}^{J-1} (2\hat{f}_j - 1)\right)' \\ &\equiv \left(\prod_{j=I-i}^{J-1} \hat{f}_j, \prod_{j=I-i}^{J-1} \hat{f}_j - 1, \prod_{j=I-i}^{J-1} (2\hat{f}_j - 1) - \prod_{j=I-i}^{J-1} \hat{f}_j\right)\end{aligned}$$

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