Market value margin via mean-variance hedging

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Motivation

- "Market consistent" valuation of liabilities in solvency context
  - Replication usually not possible
  - Cost-of-capital arguments
- Magic formula

  $V = \mathbb{E}(H) + \lambda\left[\rho(H) - \mathbb{E}(H)\right]$
  - V is value of terminal liability H
  - $\lambda \in (0, 1)$ is exogenous CoC rate
  - ρ is a regulatory risk meausure (VaR, TVaR)
- Pragmatic solution to a hard problem
  - Partial replication?
  - Multiple periods?
  - Market sensitivity?
  - Saltzmann and Wüthrich (2010), Pelsser (2010), Möhr (2011)

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What we do

- Mean-variance hedging of a terminal insurance liability
  - Discrete time / sequential regression framework (Föllmer and Schweizer, 1988; Černý and Kallsen, 2009).
  - Trade in stocks and a simple insurance derivative
- Toy model – 1 period / 1 asset
  - Derive valuation formulas that resemble CoC formulas
  - Different interpretation!
- Extensions
  - Relation to observable market prices
  - Multiple periods
  - Multiple tradeable assets

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Toy model

- Setup
  - Times $t = 0, 1$
  - Terminal liability H
  - Risk-free asset with price 1 at $t = 0$ and pay-off 1 at $t = 1$
  - Risky asset with price S_0 at $t = 0$ and pay-off S_1 at $t = 1$; excess return $X_1 = \frac{S_1}{S_0} - 1$
- Optimal investment
  - Initial endowment v, with ϑ_1 invested in risky asset
  - Value of investment portfolio at $t = 1$ is $v + \vartheta_1 X_1$
  - Market value V_0 of H identified by

$$(V_0, \xi_1) = \text{arg} \min_{(v,\vartheta_1)} \mathbb{E}\left((v + \vartheta_1 X_1 - H)^2\right)$$

$$V_0 = \mathbb{E}(H) - \frac{\text{Cov}(X_1, H)}{\text{Var}(X_1)} \mathbb{E}(X_1)$$

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The insurance derivative

- Structure of the derivative

$$S_1 = \mathbf{1}_{D_1} = \begin{cases} 1, & \text{if } D_1 \\ 0 & \text{if } D_1^c \end{cases}, \qquad D_1 = \{Z_1 \geq d_1\}$$

  - ▷ Z_1 is a trigger, d_1 a threshold
  - ▷ $p_1 = \mathbb{P}(D_1) \in (0,1)$ is probability of pay-off
  - ▷ $S_0 = q_1 \in (p_1, 1)$ is price / market-probability of pay-off
- Interpretations
  - ▷ S_1 is pay-off from an index-linked derivative triggered by D_1
  - ▷ Holder of H sponsors bond with price $1 - q_1$, paying 1 if D_1^c and 0 if D_1

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Valuation formulas

- For Z_1 such that $\mathbb{P}(Z_1 \geq d_1) = p_1$

$$V_0 = \mathbb{E}(H) + \frac{q_1 - p_1}{1 - p_1}\left[\mathbb{E}\left(H | Z_1 \geq d_1\right) - \mathbb{E}(H)\right]$$

- For indemnity trigger $Z_1 = H - \mathbb{E}(H)$

$$V_0 = \mathbb{E}(H) + \frac{q_1 - p_1}{1 - p_1}\left[\mathrm{TVaR}_{1-p_1}(H) - \mathbb{E}(H)\right]$$

- Indemnity trigger gives an upper valuation bound

$$\mathrm{TVaR}_{1-p_1}(H) \geq \mathbb{E}\left(H | Z_1 \geq d_1\right)$$

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The relation to capital

- Formulas look like CoC-based ones, but
  - ▷ $\text{TVaR}_{1-p_1}(H) - \mathbb{E}(H)$ is not capital held
  - ▷ $\frac{q_1-p_1}{1-p_1}$ is not cost-of-capital rate
- Does the trading strategy also produce capital savings?
  - ▷ Capital requirement calculated by (p.h., t.i.) risk measure ρ
  - ▷ Savings are produced when

$$\rho(H - \xi_1 X_1) \leq \rho(H)$$

  - ▷ Situations where capital saving occurs characterised for $\rho \equiv \text{VaR}_\alpha$ and $\rho \equiv \text{TVaR}_\alpha$
  - ▷ Upper bound on price q_1, which reduces in $\text{TVaR}_{1-p_1}(H - \mathbb{E}(H))$

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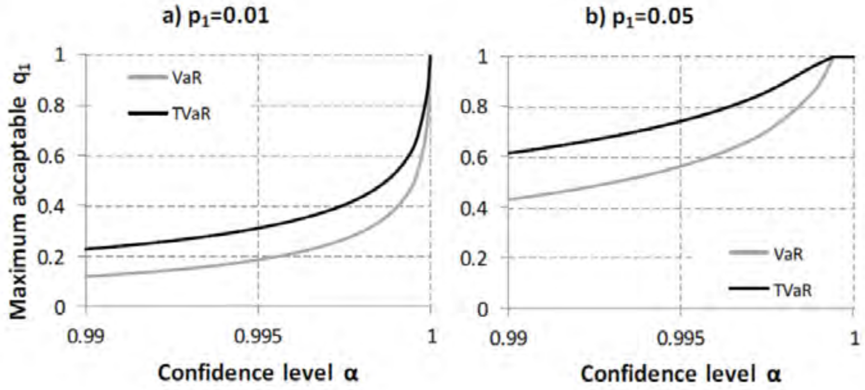
Example 1

- $H \sim \mathcal{LN}$ with $\mathbb{E}(H) = 100$, $\mathrm{Var}(H) = 20^2$
- Derivative with $p_1 = 0.01$ / $p_1 = 0.05$
- q_1s consistent with Papachristou (2011)

p_1	0.01	0.05
q_1	0.066	0.208
$\mathrm{TVaR}_{1-p_1}(H)$	166.6	148.0
$\frac{q_1-p_1}{1-p_1}$	0.057	0.167
V_0	103.8	108.0

- For VaR_α and TVaR_α, plot the maximum level of q_1 such that capital savings are produced, against α

Example 1


a) p_1=0.01
b) p_1=0.05
Maximum accaptable q_1
1
0.8
0.6
0.4
0.2
0
0.99
0.995
1
Confidence level α
VaR
TVaR

Multiple periods and assets

- Finite set of trading dates $\mathcal{T} = \{0, 1, \dots, T\}$
- Probability space $(\Omega, \mathbb{P}, \mathcal{F}, \mathbb{F})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathcal{T}}$
- $\mathcal{F}_T$-measurable liability $H \in L^2(\mathbb{P})$
- Risk free asset in each period.
- n risky assets with vector of price processes $(S_t)_{t \in \mathcal{T}}$ and of 1-period excess returns $(X_t)_{t \in \mathcal{T}}$
- Finite second moments, full rank of matrix $\mathbb{E}_t(X_{t+1} X'_{t+1})$

Multiple periods and assets

- Initial endowment v
- n-dimensional, $\mathbb{F}$-previsible trading strategy $\vartheta = (\vartheta_t)_{t \in \mathcal{T} \setminus \{0\}}$
- Portfolio value at t

$$G_t^{v,\vartheta} = v + \sum\nolimits_{k=1}^{t} \vartheta_k' X_k$$

- Optimisation problem

$$(V_0, \xi) = \text{arg}\min_{(v,\vartheta)} \mathbb{E}_0\Big((G_T^{v,\vartheta} - H)^2\Big)$$

- Solution in Černý and Kallsen (2009)

Insurance specifics

- Decomposition of the liability
  - $H = \mathbb{E}_0(H) + Y_1 + \cdots + Y_T$
  - $Y_t = \mathbb{E}_t(H) - \mathbb{E}_{t-1}(H)$ is claims development result
- Insurance derivative
  - Available at each time $t-1$ at price q_t.
  - Pay-off at time t

$$S_t^{(1)} = \mathbf{1}_{D_t} = \begin{cases} 1, & \text{if } D_t \\ 0 & \text{if } D_t^c \end{cases}, \quad \text{with} \quad D_t = \{Y_t \geq d_t\}$$

  - $\mathbb{E}_{t-1}(\mathbf{1}_{D_t}) = p_t$, such that $p_t < q_t < 1$
  - Adverse development cover

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Valuation formulas

- Assume that
    - ▷ The vectors $X_1, \dots, X_T$ are independent
    - ▷ X_t is independent of Y_{t+s} for all $0 < t < t+s \le T$
- Then

$$V_0 = \mathbb{E}_0(H) + \sum_{t=1}^{T} \mathbb{E}_0 \left(\frac{1 - a_t X_t}{1 - b_t} Y_t \right)$$

where

$$\begin{aligned} a_t &= \mathbb{E}_{t-1}(X_t') \ \left(\mathbb{E}_{t-1}(X_t X_t')\right)^{-1}, \\ b_t &= a_t \ \mathbb{E}_{t-1}(X_t) \end{aligned}$$

Valuation formulas

- We can also write

$$V_0 = \mathbb{E}_0(H) - \sum_{t=1}^{T} \frac{a_t^{(1)}}{1-b_t} \mathbb{E}_0\left(\frac{p_t}{q_t} \, \mathsf{TVaR}_{1-p_t,t-1}(Y_t) \right) - \sum_{t=1}^{T} \sum_{i=2}^{n} \frac{a_t^{(i)} \mathsf{Cov}_0(X_t^{(i)}, Y_t)}{1-b_t},$$

- Special case $n = 1$

$$V_0 = \mathbb{E}_0(H) + \sum_{t=1}^{T} \frac{q_t - p_t}{1-p_t} \mathsf{TVaR}_{1-p_t}(Y_t)$$

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Example 2

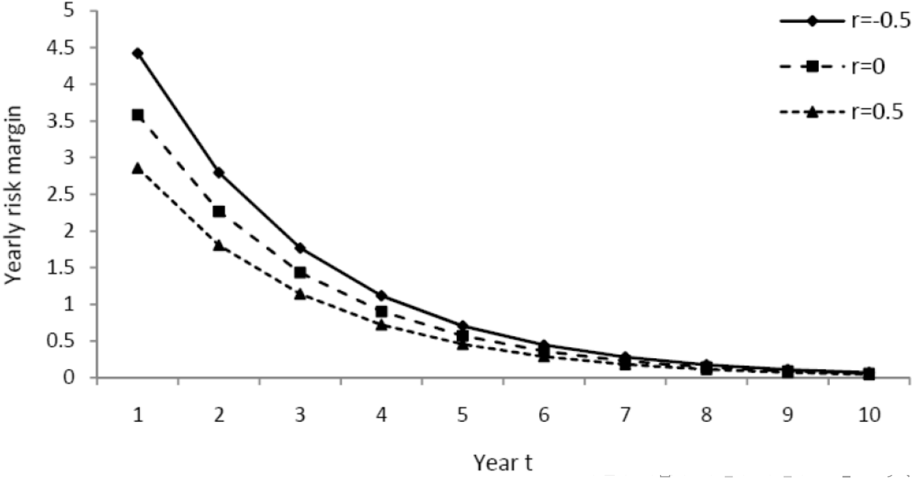
- $T = 10$, $n = 2$
- Asset/liability model

$$\begin{aligned} Y_t &= \exp\left(\mu_t + \sigma_t Z_t^{(1)}\right) - \exp\left(\mu_t + \sigma_t^2/2\right) \\ X_t^{(1)} &= \frac{\mathbf{1}_{\{Y_t \geq d_t\}}}{q_t} - 1 \\ X_t^{(2)} &= \exp\left(m + s Z_t^{(2)}\right) - 1 \end{aligned}$$

where

- ▷ $(Z_t^{(1)},\ Z_t^{(2)}) \sim \mathcal{SN}_2(r)$
- ▷ $p_t = 0.01$, $q_t = 0.066$ for all t.
- ▷ $\mu_t = 0.4586(T - t + 1)$, $\sigma_t = 0.198$ for all t
- ▷ $m = 0.15$, $s = 0.2$

Example 2


5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
Yearly risk margin
1
2
3
4
5
6
7
8
9
10
Year t
r=-0.5
r=0
r=0.5

Comments

- Capital saving?

$$\rho_{t-1}(Y_t - (G_t^{V_0,\xi(V_0)} - G_{t-1}^{V_0,\xi(V_0)})) \leq \rho_{t-1}(Y_t)$$

  - ▷ Generally not satisfied
  - ▷ Optimal investment is not considering year-on-year risk reduction
- Assumption of independence across time quite restrictive
  - ▷ Not necessary to obtain V_0...
  - ▷ ... but necessary to get pretty formulas!
  - ▷ Evaluation computationally expensive

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Example 3

- $n = 1$, $T = 2$
- $Y_t = \exp\big(\mu_t + \sigma_t Z_t\big) - \exp\big(\mu_t + \sigma_t^2/2\big)$,

  $\mu_1 = 4.586, \;\; \mu_2 = 4.127, \;\; \sigma_1 = \sigma_2 = 0.198$
- $p_1 = p_2 = 0.05$

  $$q_1 = 0.21, \; q_2 = \begin{cases} \underline{q} \leq q_1, & \text{if } Y_1 < d_1 \\ \overline{q} \geq q_1, & \text{if } Y_1 \geq d_1 \end{cases}$$
- Resulting risk margins

  $$V_0 - \mathbb{E}_0(H) = \begin{cases} 13.2, & \text{if } \overline{q}/\underline{q} = 1 \\ 14.2, & \text{if } \overline{q}/\underline{q} = 2 \\ 16.0 & \text{if } \overline{q}/\underline{q} = 4 \end{cases}$$

Conclusions

- Valuation of insurance liabilities by mean-variance hedging
  - ▷ Discrete time framework
  - ▷ Decomposition of liability into claims development results
  - ▷ Derivative on the claims development result in each period
- Valuation formulas
  - ▷ In simple cases, formulas similar to CoC
  - ▷ Extension: sensitivity to market prices, multiple assets and periods
  - ▷ Transparent price of simplicity
- Risk margin obtained is not cost of capital
  - ▷ Conceivably could add cost of capital...
  - ▷ ... as well as compensation for risk taking (Sharpe ratio)