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Actuaries Institute

Layer dependence – a measure of local dependence

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Agenda

- Dependence observed in practice
- How should we model dependence?
- **Layer dependence** and its usefulness

Common measures of dependence

Suppose u=F(x) and v=G(y) are the percentile ranks of random variables x and y, with copula C.

Spearman's rho:

$$\mathrm{cor}(u,v) = 12\int_0^1\int_0^1 C(u,v)\mathrm{d}u\mathrm{d}v - 3$$

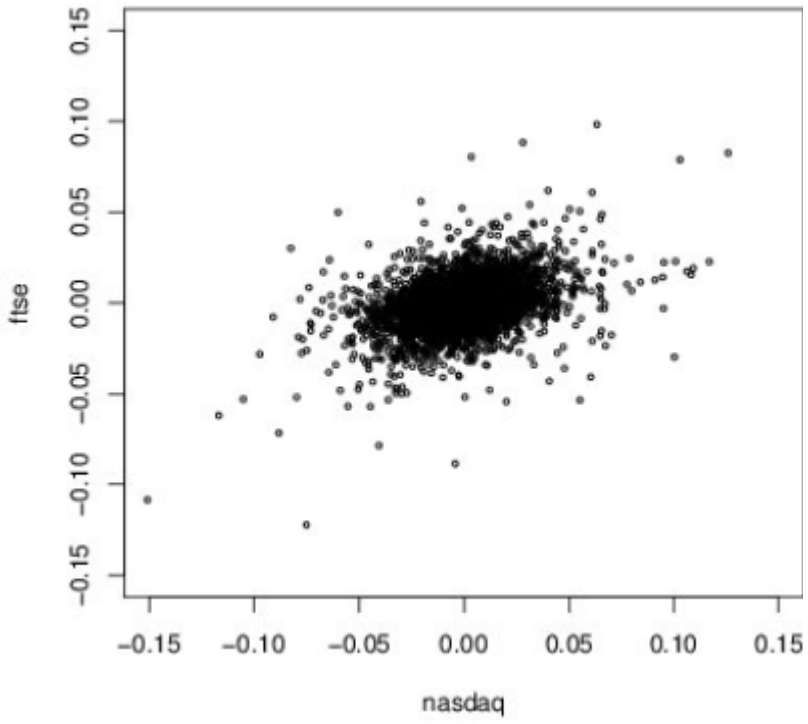
Kendall's tau:

$$4\mathrm{E}\{C(u,v)\} - 1 = 4\int_0^1\int_0^1 C(u,v)c(u,v)\mathrm{d}u\mathrm{d}v - 1$$

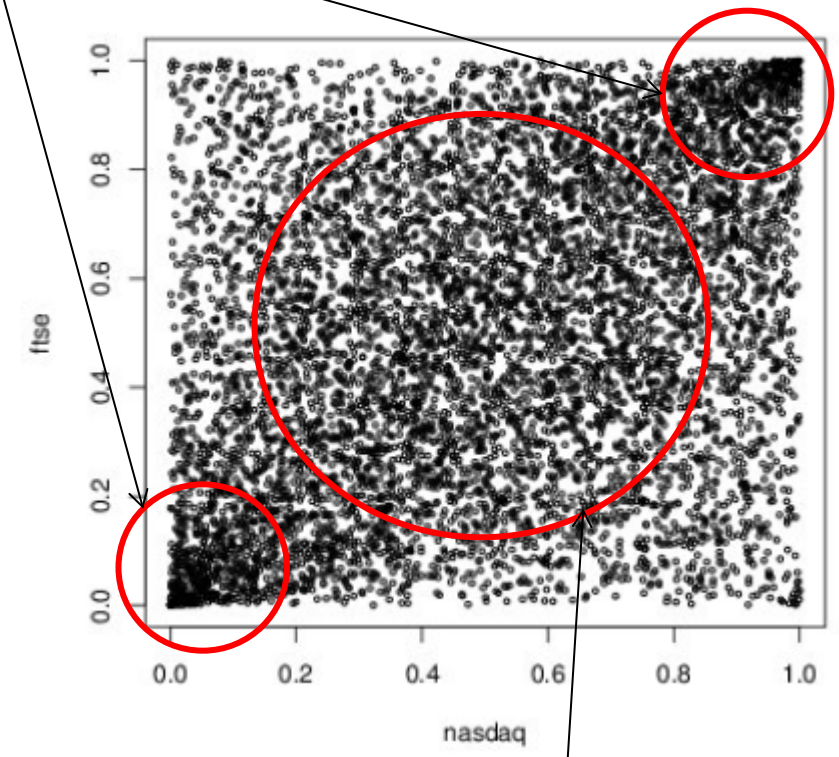
NASDAQ & FTSE returns from 1985-2015

Strong dependence (clustered)

Actual returns



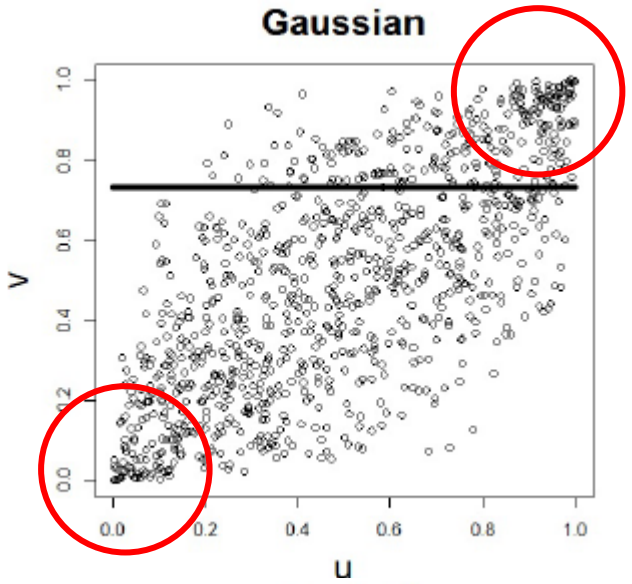
Ranked returns

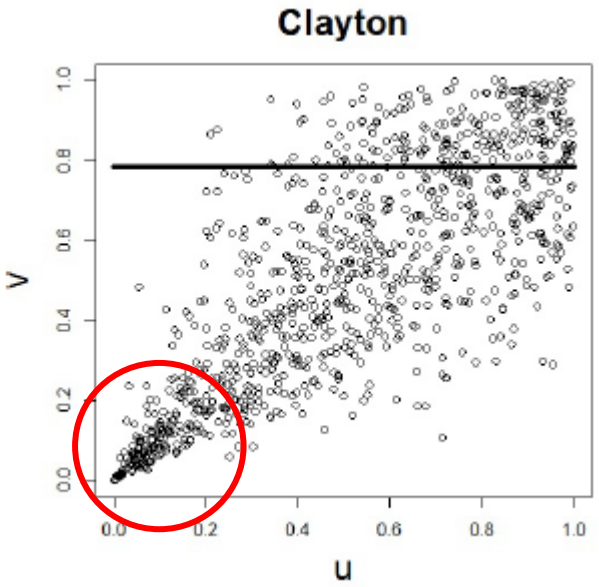


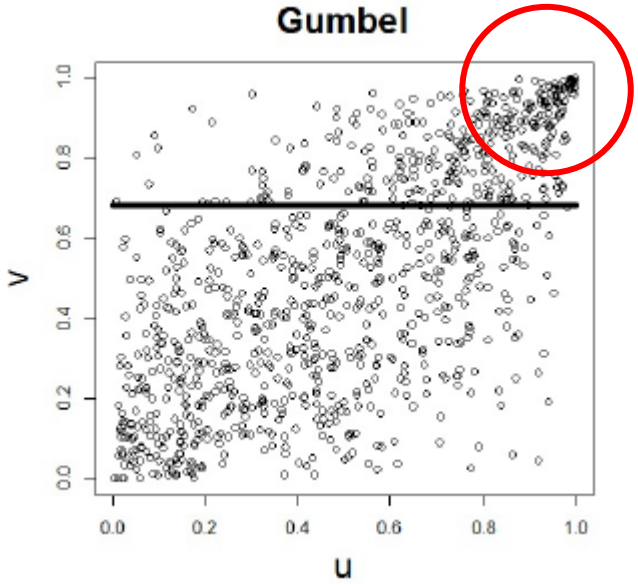
Weak dependence (dispersed)

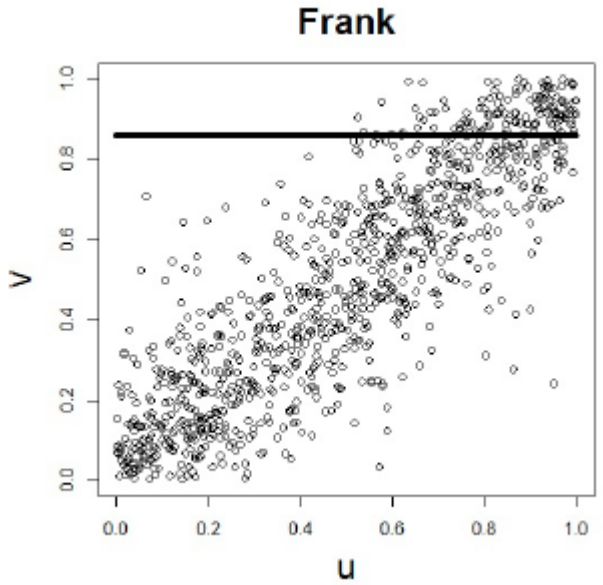
Common copulas

Flat lines represent overall correlation ρ









Observations from historical data

- Dependence varies across joint distributions of insurance and financial quantities
- Often two quantities are weakly dependent when they have moderate values, but strongly dependent when they are “in the tails”
- Think of stock market crashes, herd behaviour, natural catastrophes etc.

Implications on dependence modelling

- **Problem**: established dependence measures capture overall rather than local dependence, i.e. Spearman's rho and Kendall's tau do not tell anything about the dependence structure
- Local dependence measures are hence critical in understanding and modelling dependence

How should dependence be modelled in practice?

- **Common practice**: specify overall dependence and fit a copula which has the "right" dependence structure
- **Better practice**: specify local dependence and construct a copula which fits the specification
- **Analogy**: would you fit a probability distribution purely based on the overall mean? Or would you consider the shape of the data, e.g. volatility, skewness.

LAYER DEPENDENCE – A MEASURE OF LOCAL DEPENDENCE

Key concepts to note

- We are measuring **rank** dependence, i.e. between arbitrary uniform random variables u and v. This is consistent with the use of copulas.
- Think of excess-of-loss reinsurance **layers**, but on the percentile rank rather than absolute scale.

Layer decomposition

Write u as:

$$u = \int_0^1 l_\alpha(u)\mathrm{d}\alpha \ ,$$

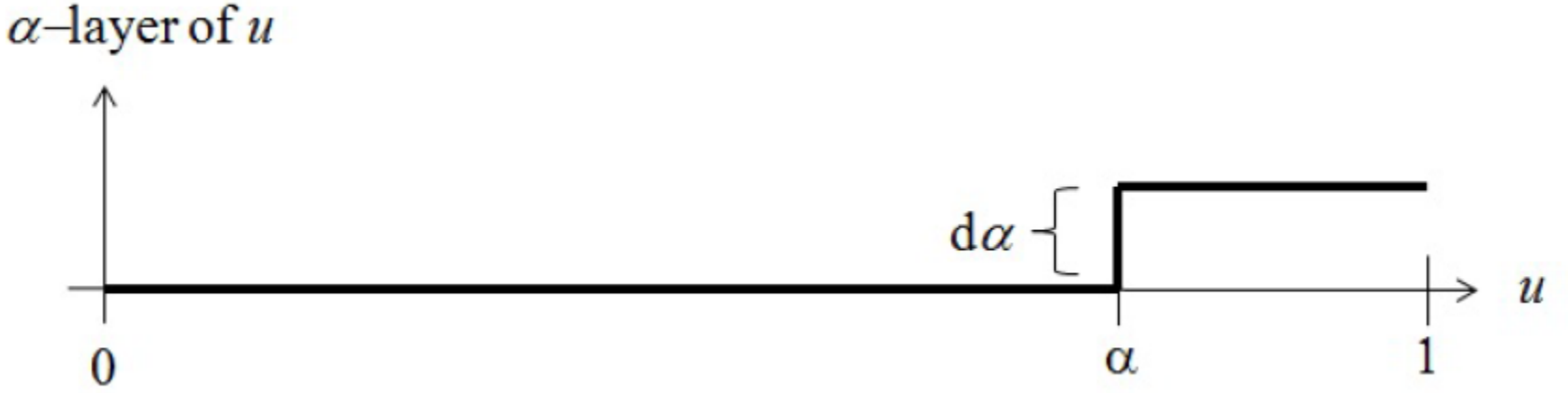
where $l_\alpha(u)$ represents an indicator function for (α>u) and is a Bernoulli rv.

Here u is decomposed into infinitesimal layers $l_\alpha(u)$ over $0<\alpha<1$.

This decomposition is commonly used in life contingencies, e.g. annuity values.

Why are layers involved?

Plot of $l_\alpha(u)$ against u:



$l_\alpha(u)$ **jumps** when u crosses α.

Therefore the layer $l_\alpha(u)$ captures movements of u **at** α. This naturally leads to the definition of layer dependence…

Definition of layer dependence

The α–layer dependence between v and u is defined as:

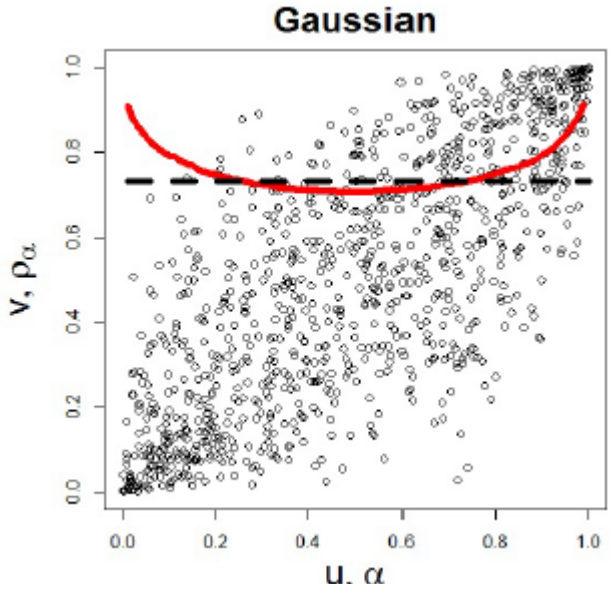
$$\rho_\alpha \equiv \frac{\mathrm{cov}\{v, l_\alpha(u)\}}{\mathrm{cov}\{u, l_\alpha(u)\}}\ , \qquad 0 \leq \alpha \leq 1\ .$$

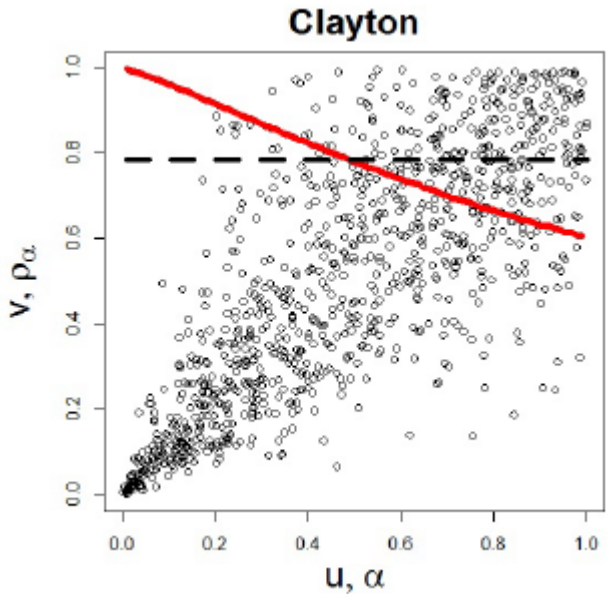
The numerator measures dependence between v and movements of u **at** α.

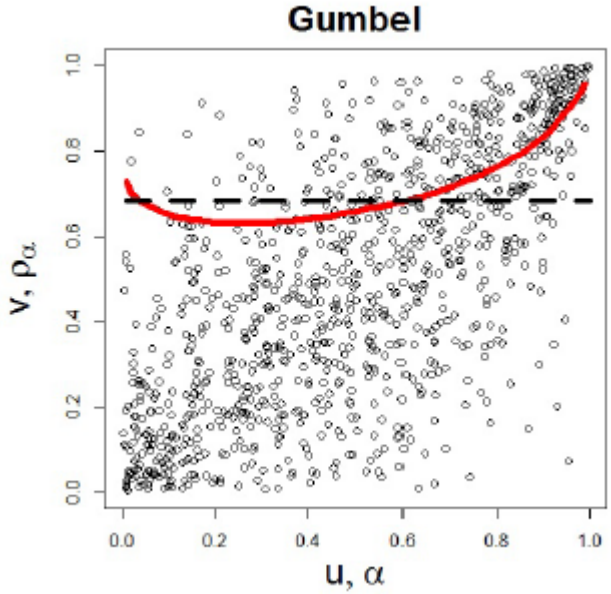
The denominator is a scaling term such that $\rho_\alpha=1$ if u=v, i.e. perfect dependence

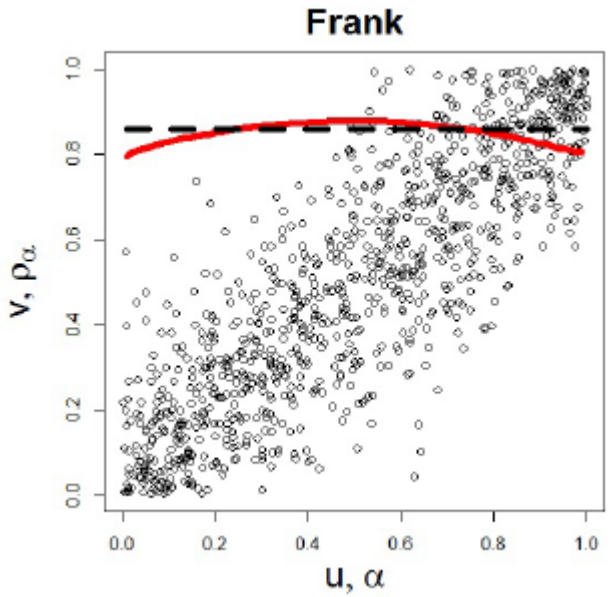
Illustration

Note that ρ_α traces local dependence (the clustering of points along 45 degree line)









Alternative expression for ρ_α

Rewriting ρ_α yield another interpretable expression:

$$\rho_\alpha = 2\{\mathrm{E}(v|u > \alpha) - \mathrm{E}(v|u \leq \alpha)\} \ ,$$

which is the "gap" between upper and lower conditional tail expectations of v where tails are defined as $u>\alpha$ and $u<\alpha$.

Hence ρ_α measures the sensitivity of v to a change in u **at** α, i.e. local dependence.

Recap: why layer dependence?

Current dependence measures such as correlation provide over-summarised information (recall examples).

On the other hand copulas provide too much information. How do you "measure" the dependence structure of a copula?

Layer dependence achieves a balance by extracting critical dependence info.

Properties of layer dependence

Layer dependence satisfies similar properties as correlation:

1. Lies between -1 and 1
2. 0 if independent, but not conversely
3. -1 and 1 is equivalent to perfect dependence
4. Symmetric over ranking order
5. Larger when dependence is stronger

Properties of layer dependence (cont)

The following properties apply over all $0<\alpha<1$:

$$-1 \leq \rho_\alpha \leq 1$$

$$C(u,v) = uv \quad \Rightarrow \quad \rho_\alpha = 0$$

$$C(u,v) = \min(u,v) \quad \Leftrightarrow \quad \rho_\alpha = 1$$

$$C(u,v) = \max(u+v-1,0) \quad \Leftrightarrow \quad \rho_\alpha = -1$$

Properties of layer dependence (cont)

The following properties apply over all $0<\alpha<1$:

$$\nu^* = 1 - \nu \quad \Rightarrow \quad \rho_\alpha^* = -\rho_\alpha$$

$$C_1 \geq C_2 \quad \Rightarrow \quad \rho_{1,\alpha} \geq \rho_{2,\alpha}$$

Link with Spearman's rho

Taking a weighted average of layer dependence ρ_α over all $0<\alpha<1$ yields Spearman's rho:

$$\int_0^1 w_\alpha \rho_\alpha \mathrm{d}\alpha = \rho \equiv \mathrm{cor}(u, v) \ , \qquad w_\alpha \equiv 6\alpha(1-\alpha) \ .$$

It makes sense to obtain **overall** dependence by averaging **local** dependence.

Food for thought:

Spearman's rho applies weights $6\alpha(1-\alpha)$ when averaging local dependence.

These weights are largest around $\alpha=0.5$ and close to zero over $\alpha=0$ and $\alpha=1$.

Does this mean that Spearman's rho (i.e. correlation) does not tell you anything at all about tail dependence?

Common copula modelling approaches

1. Select a parametric copula, e.g. Archimedean copulas
2. Fit parameters using method of moments or maximum likelihood

Q: Is the parametric form appropriate? There are only a finite number of parametric copulas available.

Q: Method of moments only fit the overall dependence, i.e. not tail dependence. What about maximum likelihood?

Applying layer dependence to copula modelling

1. Calculate layer dependence on past data to understand the dependence structure.
2. Smooth and apply expert judgement to get a layer dependence curve which characterises the desired dependence structure.
3. Fit a copula to the selected layer dependence curve. Copula may be parametric or semi-parametric.