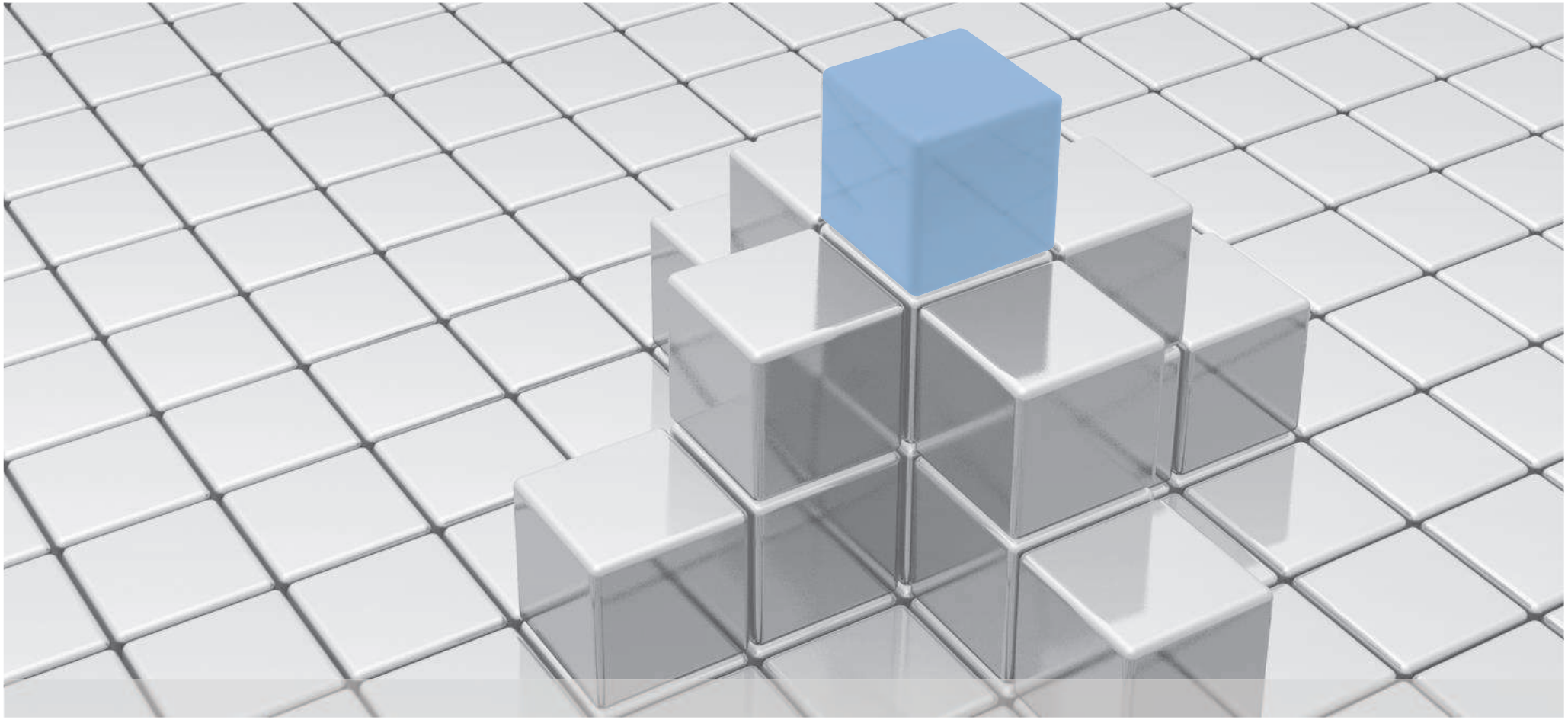

BINARY EVENTS LOADING FOR SOLVENCY II TECHNICAL PROVISIONS: PRACTICAL APPROXIMATIONS

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Motivation

Under the upcoming Solvency II insurers are required to allow for all possible events when setting their technical provisions, including those that may not have been historically realised, i.e.

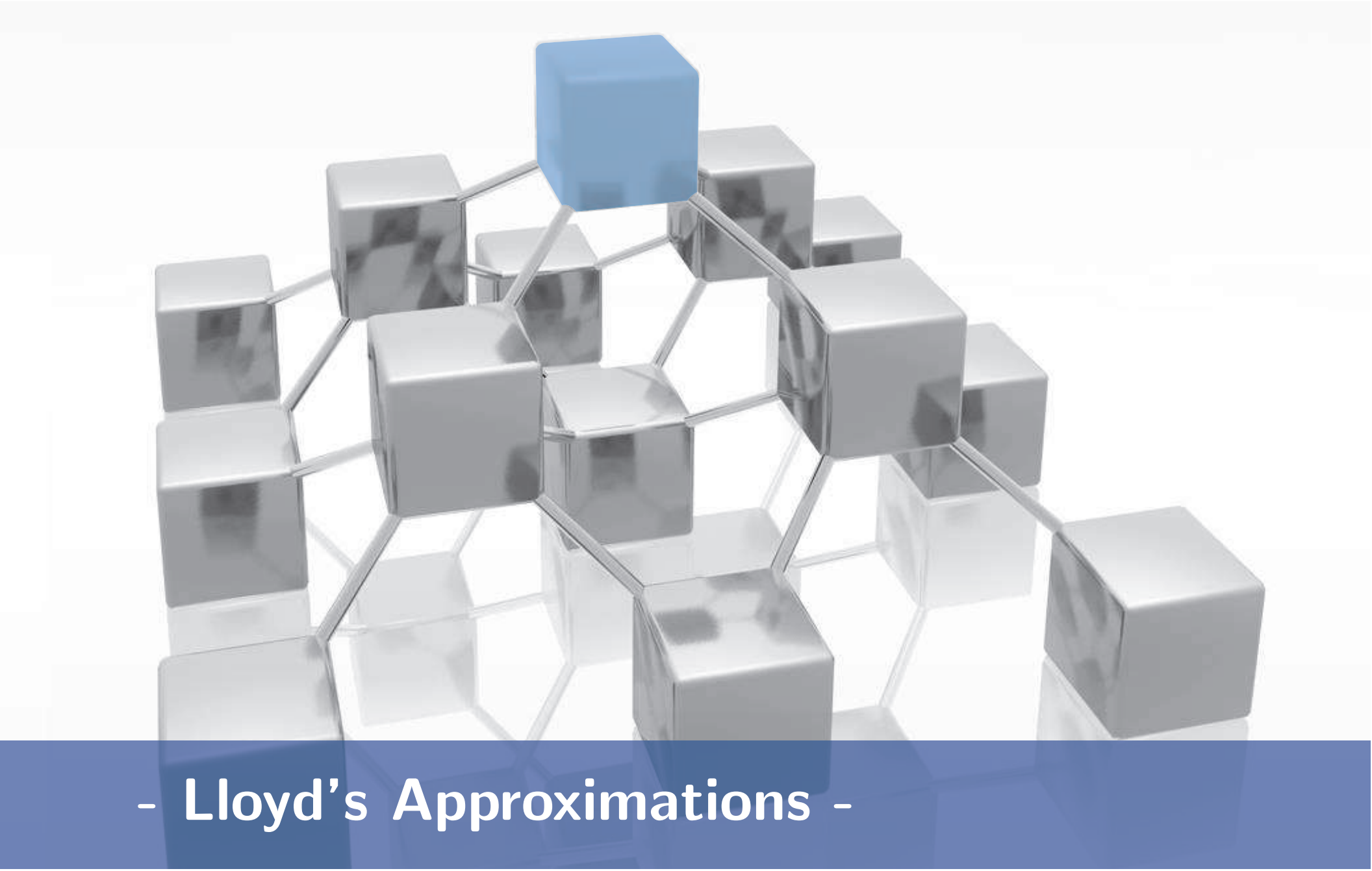
- *Binary Events* or *Events Not In Data (ENID)*

There are many ENID loading approaches, and it's up to insurers to decide which one suits best.

Lloyd's approximations: Lloyd's Technical Provisions Guidance (2011) recommends using the *Truncated Statistical Approach*, however does not provide explicit analytical formulae for calculating the uplift. The industry is in favour of two particular analytical approximations of the reserve mean load assuming log-normality of reserve risk profile.

This research

1. examines the quality of Lloyd's approximations; and
2. proposes a new distribution-free approach to estimating ENID load



- Lloyd's Approximations -

Background

The **Truncated Statistical Distribution approach** defines the reserve uplift factor as

the ratio of the 'true mean' to the 'mean only including realistically foreseeable events'

- 'realistically foreseeable events' - loss events with a return period of up to Y years
- the true (untruncated) reserve value is X and its distribution has a parametric form F_X
- the reserve values based on the 'realistically foreseeable events' are drawn from the truncated reserve distribution, or the distribution of conditional reserve value X given $X \leq F_X^{-1}(p)$, where $p = 1 - 1/Y$
- the mean and variability (CoV) of reserve based on realistically foreseeable events only are available to the actuarial function.

Lloyd's Approximations

Two 'Lloyd's approximations' of the load of reserve mean, $\mathbf{Mean\_Load}$, assuming **log-normality** of the true reserve distribution (here $\mathbf{Mean\_Load}$ is simply **uplift factor** minus 1):

Lloyd's Approximation 1:

$$\frac{p}{\Phi\left(\Phi^{-1}(p) - \sqrt{\ln\left(\mathrm{CoV_{tr}}^2 + 1\right)}\right)} - 1 \tag{1}$$

Lloyd's Approximation 2:

$$\frac{1}{\Phi\left(\Phi^{-1}(p) - \sqrt{\ln\left(\mathrm{CoV_{tr}}^2 + 1\right)}\right)} - 1 \tag{2}$$

where $\mathrm{CoV_{tr}}$ is the coefficient of variation of the reserve based on the truncated set of loss data representing realistically foreseeable events with the return period of up to $Y = 1/(1-p)$ years.

- The two approximations, whilst being similar, could produce noticeably different results (both differ by factor p)
- Quality of approximations?

Derivation of Lloyd's Approximations (1)

Derivation from first principles. The k-th non-central moment of X_{obs} is the k-th truncated non-central moment of X (log-normally distributed) which is equal to:

$$\begin{aligned} m_{tr}^{(k)} = \mathbb{E}[X_{obs}^k] &= \mathbb{E}[X^k | X \leq b] \\ &= m^{(k)} \frac{\Phi\left(\frac{\ln b - \mu}{\sigma} - k\sigma\right)}{\Phi\left(\frac{\ln b - \mu}{\sigma}\right)} \\ &= m^{(k)} \frac{\Phi\left(\Phi^{-1}(p) - k\sigma\right)}{p} \end{aligned} \tag{3}$$

Equation 3 is used to derive the load for reserve Mean and CoV.

Derivation of Lloyd's Approximations (2)

Reserve Mean load:

$$\begin{aligned} m_{tr}^{(1)} &= \mathbb{E}[X|X \leq b] \\ &= m^{(1)} \frac{\Phi\left(\Phi^{-1}(p) - \sigma\right)}{p} = m^{(1)} \times \alpha, \end{aligned} \tag{4}$$

and thus

$$\mathbf{Mean\_Load} = \frac{m^{(1)}}{m_{tr}^{(1)}} - 1 = \frac{1}{\alpha} - 1,$$

where $\alpha = \frac{\Phi\left(\Phi^{-1}(p)-\sigma\right)}{p} \in (0,1)$ for $p \in [0.95, 1)$, and tends to 1 as p goes to 1.

Derivation of Lloyd's Approximations (3)

Reserve CoV load:

$$
\begin{aligned}
{\mathrm{CoV}_{\mathrm{tr}}}^2 &= \frac{m_{tr}^{(2)} - \left(m_{tr}^{(1)}\right)^2}{\left(m_{tr}^{(1)}\right)^2} = \frac{m_{tr}^{(2)}}{\left(m_{tr}^{(1)}\right)^2} - 1 \\
&\overset{(4)\&\ (3)}{=} \frac{\left(m^{(1)}\right)^2 e^{\sigma^2}\, \Phi\left(\Phi^{-1}(p) - 2\sigma\right)/p}{\left(m^{(1)}\right)^2 \alpha^2} - 1 \\
&= e^{\sigma^2}\, \frac{\Phi\left(\Phi^{-1}(p) - 2\sigma\right)}{p\,\alpha^2} - 1 \qquad (5) \\
&\approx e^{\sigma^2} - 1, \qquad (6)
\end{aligned}
$$

as $\frac{\Phi\left(\Phi^{-1}(p) - 2\sigma\right)}{p\,\alpha^2} \to 1$, when $p \to 1$.

Formally, $\exists \theta_0 \in (0,1)$ such that $\forall \theta \in (0, \theta_0)$:

$$
\frac{\Phi\left(\Phi^{-1}(p) - 2\sigma\right)}{p\,\alpha^2} = 1 - o\left((1-p)^{\theta}\right).
$$

Derivation of Lloyd's Approximations (4)

From here we get the estimate $\widehat{\sigma} = \sqrt{\ln\left(\mathrm{CoV_{tr}}^2 + 1\right)}$, and use it further to approximate $\mathbf{Mean\_Load}$:

$$\begin{aligned}
\textbf{Approximation 1} &= \frac{1}{\alpha} - 1, \\
&= \frac{p}{\Phi\left(\Phi^{-1}(p) - \widehat{\sigma}\right)} - 1, \\
&= \frac{p}{\Phi\left(\Phi^{-1}(p) - \sqrt{\ln\left(\mathrm{CoV_{tr}}^2 + 1\right)}\right)} - 1.
\end{aligned}$$

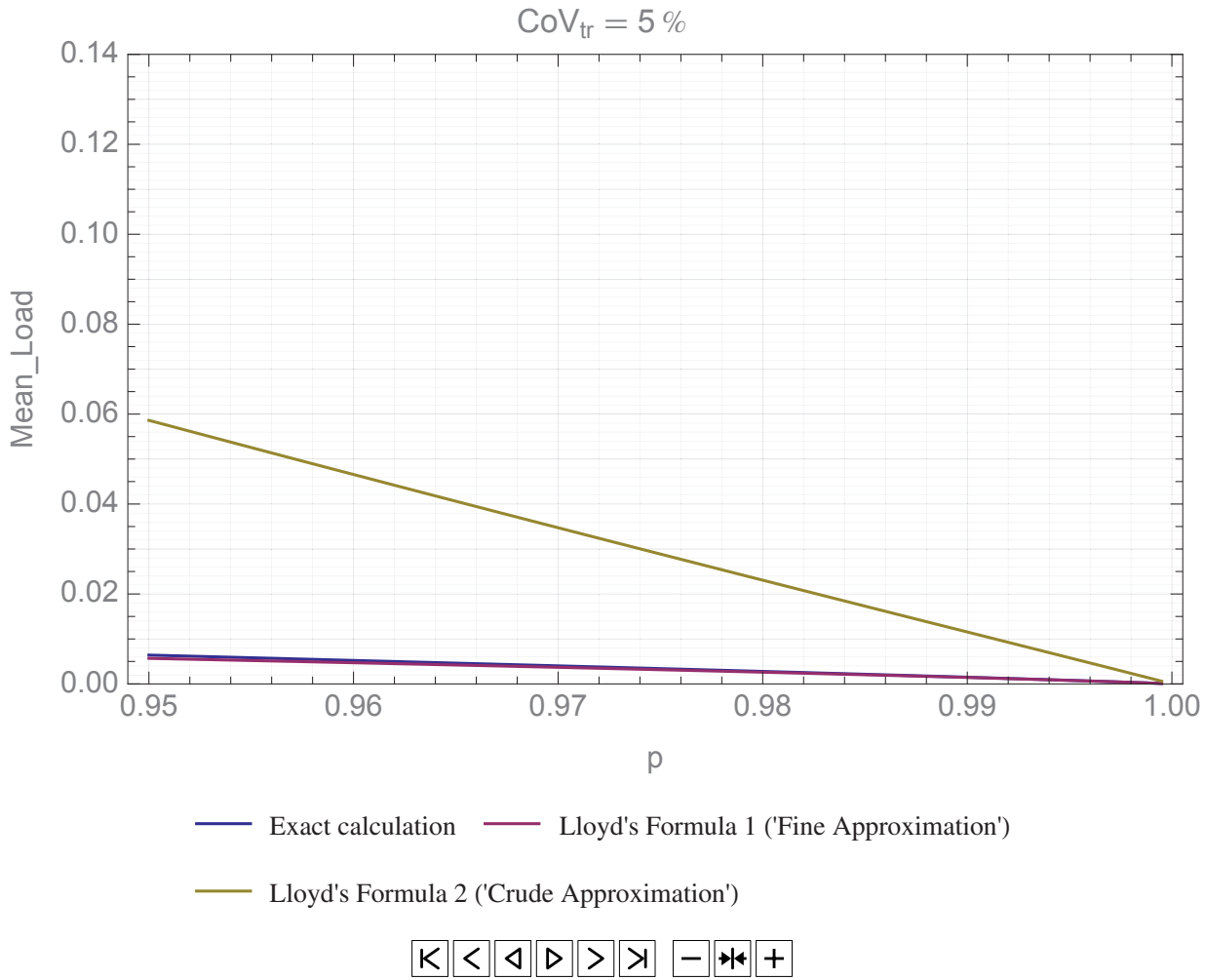
Further approximation is then obtained from **Approximation 1** by assuming $p \approx 1$, in which case we have a crude approximation

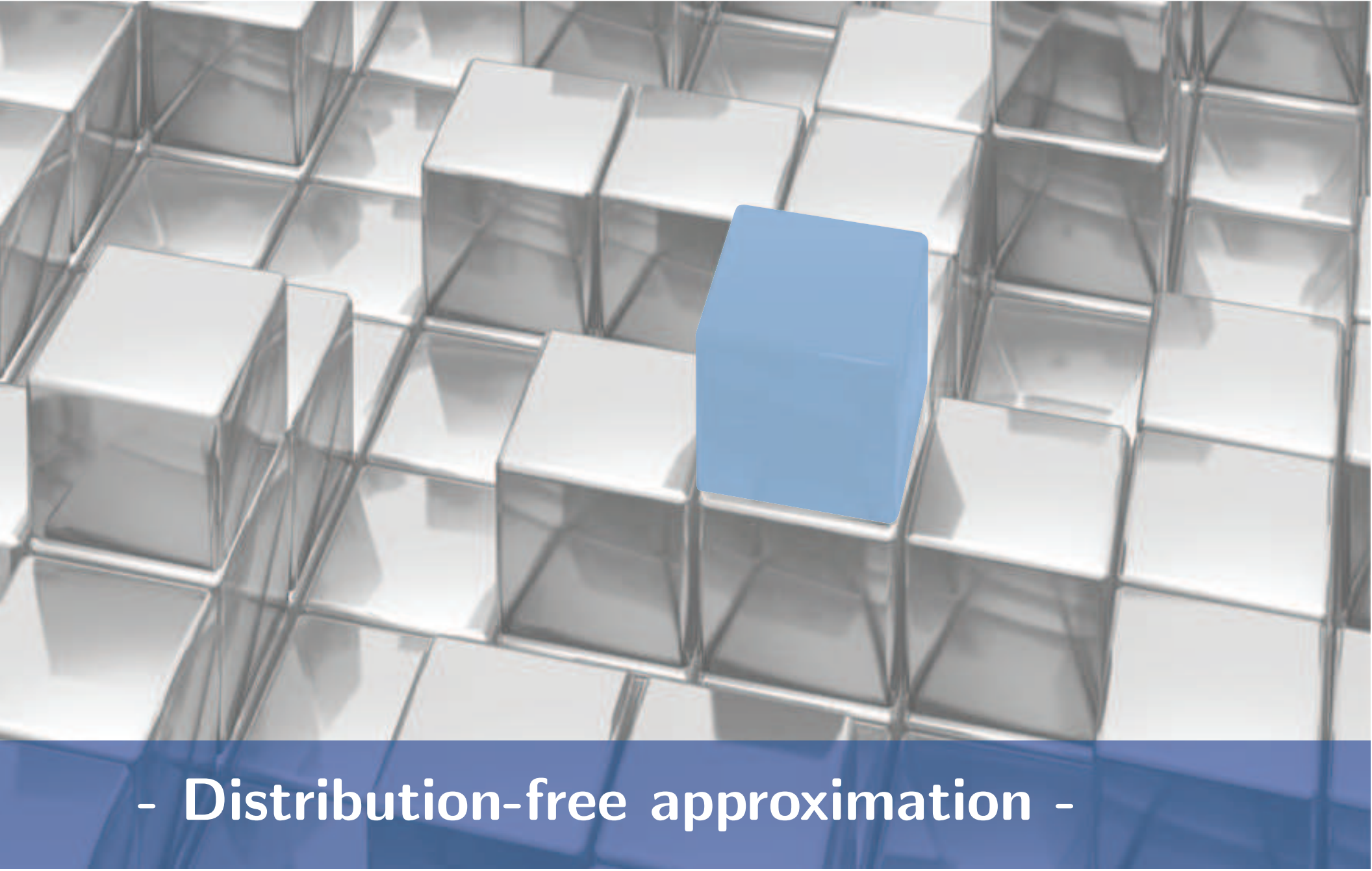
$$\textbf{Approximation 2} = \frac{1}{\Phi\left(\Phi^{-1}(p) - \sqrt{\ln\left(\mathrm{CoV_{tr}}^2 + 1\right)}\right)} - 1.$$

Quality of Lloyd's Approximations

Approximation 1 is generally better than Approximation 2.

Functional relationship between $\mathbf{Mean\_Load}$ and p under varying value of $\mathrm{CoV_{tr}}$.





- Distribution-free approximation -

Setting the problem ...

- **Key assumption:** the true unknown distribution of reserve X is assumed to be of $\mathcal{SSP}$ type, and thus the unknown true value of skewness is defined by

$$\gamma\left(\mathrm{CoV}\right) = SC\left(\mathrm{CoV}\right) \cdot \mathrm{CoV}. \tag{7}$$

- **Goal:** find the way of estimating $\mathbf{Mean\_Load} = \frac{m}{m_{tr}} - 1$ using only the following information

  1. CoV_{tr} – 'observable' volatility of reserve X; and

  2. SC ratio of reserve distribution (one of the key characteristics of a $\mathcal{SSP}$ type of distribution)

- **Approximations are tabulated:** by CoV_{tr}, SC and p. Estimate adjustments can be applied if possible?

Getting the distribution-free approximation (1)

Key steps:

1. $X = m\left(1 + \mathrm{CoV} \cdot \widetilde{X}\right)$, where $\widetilde{X} = \frac{X-m}{s}$, and also

$$\mathrm{VaR}_p\left(X\right) = m\left(1 + \mathrm{CoV} \cdot \mathrm{VaR}_p\left(\widetilde{X}\right)\right)$$

2. **The truncated mean** of reserve X is then

$$\begin{aligned} m_{tr} &= \mathbb{E}[X \,|\, X \leq \mathrm{VaR}_p\left(X\right)] \\ &= m\left(1 + \mathrm{CoV} \cdot \mathbb{E}\left[\widetilde{X} \,|\, \widetilde{X} \leq \mathrm{VaR}_p\left(\widetilde{X}\right)\right]\right), \end{aligned}$$

from where the ENID uplift factor for reserve mean is defined as

$$\boxed{\frac{m}{m_{tr}} = \frac{1}{1 + \mathrm{CoV} \cdot \mathbb{E}\left[\widetilde{X} \,|\, \widetilde{X} \leq \mathrm{VaR}_p\left(\widetilde{X}\right)\right]}} \tag{8}$$

Getting the distribution-free approximation (2)

3. **The truncated variance** of X is calculated as follows

$$\begin{aligned} s_{tr}^2 &= \mathbb{E}\left[(X - m_{tr})^2 \mid X \leq \mathrm{VaR}_p(X)\right] \\ &= m^2 \,\mathrm{CoV}^2 \left(\mathbb{E}\left[\widetilde{X}^2 \mid \widetilde{X} \leq \widetilde{b}\right] - \mathbb{E}^2\left[\widetilde{X} \mid \widetilde{X} \leq \widetilde{b}\right]\right), \end{aligned} \tag{9}$$

where $\widetilde{b} = \mathrm{VaR}_p\left(\widetilde{X}\right)$. By multiplying corresponding sides of Equation 9 and squared Equation 8 and then dividing both sides of the resulting equation by m^2, we obtain the following formula for truncated variance:

$$\boxed{\mathrm{CoV}_{tr}^2 = \frac{\mathrm{CoV}^2 \cdot \left(\mathbb{E}\left[\widetilde{X}^2 \mid \widetilde{X} \leq \widetilde{b}\right] - \mathbb{E}^2\left[\widetilde{X} \mid \widetilde{X} \leq \widetilde{b}\right]\right)}{\left(1 + \mathrm{CoV} \cdot \mathbb{E}\left[\widetilde{X} \mid \widetilde{X} \leq \widetilde{b}\right]\right)^2}} \tag{10}$$

Getting the distribution-free approximation (3)

4. $\widetilde{X}$ **is assumed to be approximated by the Fleishman quadratic polynomial** of a standard normal random variable $Z \sim \mathcal{N}(0,1)$:

$$\widetilde{X} \overset{d}{\approx} P_2(Z) = a_1 Z + a_2(Z^2 - 1), \tag{11}$$

where the Fleishman coefficients a_1 and a_2 are calibrated so that $P_2(Z)$ has unit variance and its skewness is equal to γ - skewness of X:

$$\begin{cases} 1 &= a_1^2 + 2a_2^2, \\ \gamma &= 6a_1^2 a_2 + 8a_2^3 \end{cases} \tag{12}$$

5. The p-quantile of random variable $\widetilde{X}$ **is assumed to be approximated by the Normal Power approximation**:

$$\widetilde{b} = \mathrm{VaR}_p\left(\widetilde{X}\right) \approx z_p + \gamma\,\frac{z_p^2 - 1}{6}, \tag{13}$$

where $z_p = \mathrm{VaR}_p\left(Z\right)$.

Getting the distribution-free approximation (4)

Further analytical transformations:

6. Calibrate Fleishman coefficients a_1 and a_2, and express them as functions of CoV;

7. Express the n-th truncated moment $\mathbb{E}\left[\widetilde{X}^n \,|\, \widetilde{X} \leq \widetilde{b}\right]$ for $n = 1, 2$ as a function of CoV by:

   - finding the equivalent probability condition of $\left\{\widetilde{X} \leq \widetilde{b}\right\}$ defined through the random variable Z – this would likely to be in the following form $\{c \leq Z \leq d\}$ with c and d being functions of CoV; and then

   - calculating the n-th truncated moment of the standard normal random variable Z, i.e. $\mathbb{E}\left[Z^n \,|\, c \leq Z \leq d\right]$ for $n = 1, ..., 4$.

Calibration of Fleishman coefficients (Step 6.)

Requires solving a cubic equation:

$$\begin{cases} a_1 &= \sqrt{1-2a_2^2}, \\ \gamma &= 6a_2 - 4a_2^3 \end{cases} \tag{14}$$

It is feasible, when $|a_2| \leq \frac{1}{\sqrt{2}}$ and thus $0 \leq \gamma \leq 2\sqrt{2}$, hence the Fleishman approximation is not suitable for extremely volatile and skewed reserve risk profiles (e.g. those that are adhering to Inverse-Gamma parametric distribution and have CoV above 50%).

It can be shown that after taking into account the lower and upper bounds of a_2 there is only one real root, which is positive and equal to:

$$a_2\left(\mathrm{CoV}\right) = \sqrt{2}\cos\left(\frac{\varphi(\mathrm{CoV})}{3} + \frac{4\pi}{3}\right), \tag{15}$$

where $\varphi\left(\mathrm{CoV}\right) = \arccos\left(-\frac{\gamma(\mathrm{CoV})}{2\sqrt{2}}\right)$.

Finding equivalent probability condition (Step 7.)

$\left\{\widetilde{X} \leq \widetilde{b}\right\}$ is equivalent to solving the following quadratic inequality for Z:

$$a_2 Z^2 + a_1 Z - \left(a_2 + \widetilde{b}\right) \leq 0, \tag{16}$$

where $\widetilde{b}$ is defined in Equation 13 and is a positive function of CoV for $p > 0.85$ and $\forall\ \gamma > 0$.

Given $a_2 > 0$, the solution to quadratic inequality (16) is the interval $c \leq Z \leq d$, where

$$c = \frac{-a_1 - \sqrt{a_1^2 + 4a_2(a_2 + \widetilde{b})}}{2a_2}, \tag{17}$$

$$d = \frac{-a_1 + \sqrt{a_1^2 + 4a_2(a_2 + \widetilde{b})}}{2a_2}. \tag{18}$$

Moreover, $c < 0 < d$ as $a_1 > 0$.

Getting the n-th truncated moments of Z (Step 7.)

Let us further denote the n-th truncated moment of Z by

$$I_n \quad = \quad \mathbb{E}\left[Z^n \,|\, c \leq Z \leq d\right], \, n \geq 0. \tag{19}$$

Then using the Fleishman approximation of $\widetilde{X}$ (11) and also (17) and (18), we obtain

$$\begin{aligned}
\mathbb{E}\left[\widetilde{X} \,|\, \widetilde{X} \leq \widetilde{b}\right] &= a_2 I_2 + a_1 I_1 - a_2 I_0, \\
\mathbb{E}\left[\widetilde{X}^2 \,|\, \widetilde{X} \leq \widetilde{b}\right] &= a_2^2 I_4 + 2a_1 a_2 I_3 + \left(1 - 4a_2^2\right) I_2 \\
&\quad -2a_1 a_2 I_1 + a_2^2 I_0.
\end{aligned}$$

The n-th truncated moment of Z in (19) can be computed iteratively using the following formula

$$\begin{cases} I_n &= -\frac{d^{n-1}\varphi(d) - c^{n-1}\varphi(c)}{\Phi(d) - \Phi(c)} + (n-1) I_{n-2}, \text{ with} \\ I_1 &= -\frac{\varphi(d) - \varphi(c)}{\Phi(d) - \Phi(c)}, \text{ and } I_0 = 1. \end{cases}$$

Getting the final approximation

We conclude that both $\mathbb{E}\left[\widetilde{X} \mid \widetilde{X} \leq \widetilde{b}\right]$ and $\mathbb{E}\left[\widetilde{X}^2 \mid \widetilde{X} \leq \widetilde{b}\right]$ are analytical functions of CoV, as

- I_n is a function of c and d (see (19));
- both c and d are functions of a_1, a_2 and $\widetilde{b}$ (see (17) and (18)); and finally
- a_1, a_2 and $\widetilde{b}$ are analytical functions of $\gamma(\mathrm{CoV})$.

The following equation is then numerically solved for CoV:

$$\mathrm{CoV}_{tr}^2 = \frac{\mathrm{CoV}^2 \cdot \left(\mathbb{E}\left[\widetilde{X}^2 \mid \widetilde{X} \leq \widetilde{b}\right] - \mathbb{E}^2\left[\widetilde{X} \mid \widetilde{X} \leq \widetilde{b}\right]\right)}{\left(1 + \mathrm{CoV} \cdot \mathbb{E}\left[\widetilde{X} \mid \widetilde{X} \leq \widetilde{b}\right]\right)^2} \tag{20}$$

The derived ultimate estimate $\widehat{\mathrm{CoV}}$ is then used to estimate the ENID load.

Quality of the distribution-free approximation ...

... is generally good across all $\mathcal{SSP}$ distributions in the range between Gamma and Inverse-Gamma. Sample output:

Table 4: *Approximated* **Mean_Load** *(in %) assuming Log-Normal distribution and* CoV $\leq$ 50% *(i.e.* $3 < SC \leq 3.25$*).*

	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
$\frac{\text{CoV}_{tr}}{}$ =									
10%	1.346	1.221	1.095	0.969	0.841	0.712	0.582	0.448	0.311
15%	2.120	1.921	1.723	1.523	1.322	1.119	0.913	0.704	0.488
20%	2.960	2.681	2.402	2.122	1.841	1.557	1.270	0.978	0.678
25%	3.869	3.501	3.134	2.766	2.397	2.026	1.652	1.271	0.881
30%	4.850	4.384	3.919	3.456	2.992	2.526	2.058	1.582	1.096
35%	5.908	5.333	4.762	4.194	3.627	3.059	2.489	1.913	1.324
40%	7.047	6.352	5.664	4.982	4.303	3.626	2.947	2.263	1.565
45%	8.263	7.439	6.625	5.820	5.022	4.227	3.433	2.634	1.821
50%	9.527	8.573	7.632	6.701	5.778	4.861	3.946	3.027	2.094

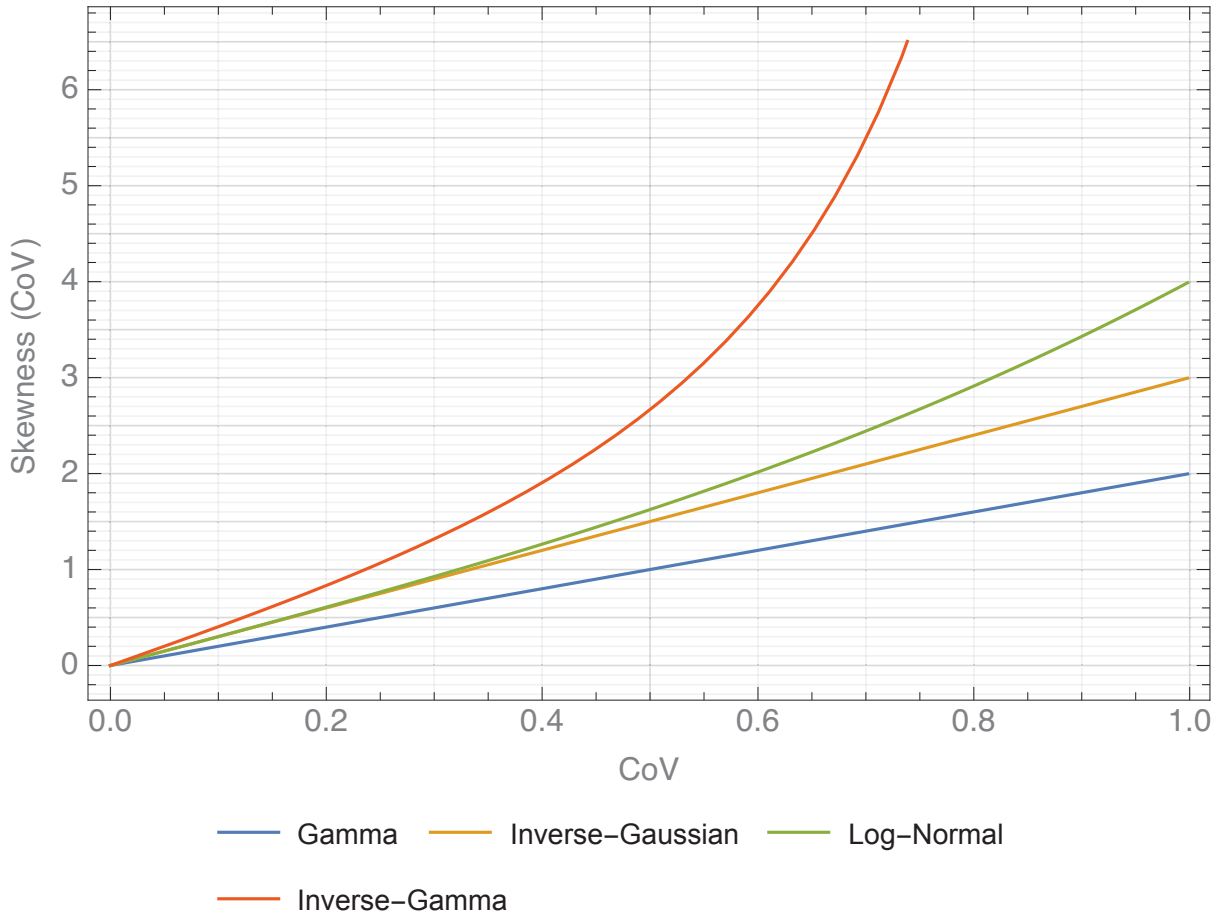
Table 5: *Exact value of* **Mean_Load** *(in %) assuming Log-Normal distribution and* CoV $\leq$ 50% *(i.e.* $3 < SC \leq 3.25$*).*

	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
$\frac{\text{CoV}_{tr}}{}$ =									
10%	1.362	1.236	1.109	0.982	0.853	0.723	0.591	0.456	0.317
15%	2.172	1.970	1.768	1.565	1.360	1.153	0.943	0.728	0.506
20%	3.079	2.793	2.506	2.218	1.927	1.634	1.337	1.033	0.720
25%	4.093	3.711	3.329	2.946	2.560	2.171	1.776	1.373	0.958
30%	5.223	4.735	4.246	3.756	3.263	2.767	2.264	1.752	1.223
35%	6.481	5.872	5.263	4.654	4.043	3.427	2.805	2.171	1.517
40%	7.877	7.133	6.391	5.648	4.904	4.156	3.401	2.632	1.841
45%	9.425	8.529	7.636	6.745	5.853	4.958	4.056	3.139	2.197
50%	11.138	10.071	9.010	7.952	6.897	5.839	4.774	3.695	2.586

Location of reserve risk profile ...

... with respect to four known SSP parametric distributions:

Figure 1: *Skewness as a function of CoV for the four parametric distributions.*



Tabulated results

ENID load approximations can be pre-computed and tabulated by CoV_{tr}, SC and p. Sample output:

Table 24: **Mean_Load** *approximation (in %) under* $\mathrm{CoV}_{\mathrm{tr}} = 30\%$.

	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
SC =									
2.0	4.415	4.001	3.585	3.169	2.749	2.327	1.899	1.463	1.014
2.2	4.492	4.069	3.646	3.221	2.794	2.364	1.928	1.485	1.030
2.4	4.570	4.138	3.706	3.273	2.838	2.400	1.957	1.507	1.045
2.6	4.647	4.206	3.765	3.324	2.881	2.436	1.986	1.529	1.060
2.8	4.724	4.274	3.825	3.375	2.925	2.472	2.015	1.550	1.074
3.0	4.802	4.342	3.884	3.426	2.968	2.507	2.043	1.572	1.089
3.2	4.879	4.410	3.943	3.477	3.010	2.542	2.070	1.592	1.103
3.4	4.957	4.479	4.002	3.527	3.053	2.577	2.098	1.613	1.117
3.6	5.035	4.547	4.061	3.578	3.095	2.611	2.125	1.634	1.131
3.8	5.113	4.615	4.120	3.628	3.137	2.646	2.152	1.654	1.145
4.0	5.191	4.683	4.179	3.678	3.179	2.680	2.179	1.674	1.158
4.2	5.269	4.751	4.237	3.727	3.220	2.714	2.206	1.694	1.172
4.4	5.346	4.818	4.295	3.777	3.261	2.747	2.233	1.714	1.186
4.6	5.422	4.884	4.352	3.825	3.302	2.781	2.259	1.734	1.199
4.8	5.497	4.949	4.409	3.873	3.342	2.814	2.285	1.754	1.213
5.0	5.568	5.012	4.463	3.920	3.382	2.846	2.311	1.773	1.227
5.2	5.636	5.073	4.516	3.966	3.421	2.878	2.337	1.793	1.240

Practical implementation

For given CoV_{tr}, SC and p

- read (grid search) the tabulated approximation;
- compute the correction factor by
  - *locating the reserve risk profile with respect to known SSP distributions* – by comparing its SC ratio to those of the known parametric SSP distributions;
  - interpolate the correction factor between the parametric distributions adjacent to the reserve risk profile; and
- use it to adjust the initial distribution-free approximation of ENID load.

Thank You

Q & A