ASTIN, AFIR/ERM and IACA Colloquia
Innovation & Invention
23-27 August 2015 | Amora Hotel Jamison Sydney

Actuaries Institute

Binary Events Loading for Solvency II Technical Provisions: practical approximations

Prepared by Yuriy Krvavych

Presented to the Actuaries Institute
ASTIN, AFIR/ERM and IACA Colloquia
23-27 August 2015
Sydney

Institute of Actuaries of Australia
ABN 69 000 423 656
Level 2, 50 Carrington Street, Sydney NSW Australia 2000
t +61 (0) 2 9233 3466 **f** +61 (0) 2 9233 3446
e actuaries@actuaries.asn.au **w** www.actuaries.asn.au

BINARY EVENTS LOADING FOR SOLVENCY II TECHNICAL PROVISIONS: PRACTICAL APPROXIMATIONS

YURIY KRVAVYCH

ASTIN colloquium

sydney, 23-27 august, 2015

BINARY EVENTS LOADING FOR SOLVENCY II TECHNICAL PROVISIONS: PRACTICAL APPROXIMATIONS

YURIY KRVAVYCH*

Under the Solvency II Directive insurers are required to allow for all possible events when setting their technical provisions, including those that may not have been historically realised before. Such events not presented in a set of observable historical loss data are often called Binary Events. Here, the term 'Binary' is traditionally used to define loss-generating events with low frequency and high severity impact, whereas an alternative name, Events Not In Data (ENID), may also be used to denote a much broader set of unobservable loss events. In its Technical Provisions Guidance issued in March 2011, whilst suggesting several approaches to allowing for Binary Events, Lloyd's encourages actuarial functions to use one specific approach that offers uplifting reserve best estimates to allow for a limited range of historical (observable) data. This approach is called the Truncated Statistical Distribution approach. Although the Lloyd's guidance does not provide any explicit analytical formulae for calculating the uplift factor, it seems the industry is in favour of two particular analytical approximations of the load of reserve best estimate, assuming log-normality of the true distribution of reserves. This paper examines the quality of the two Lloyd's approximations of Binary Events loading for technical provisions, and also presents a distribution-free approach to estimating ENID load for reserve mean and variability using the following key determinants of reserve risk profile:

- the level of variability measured by Coefficient of Variation (CoV); and
- the degree of Skewness per unit of CoV.

Keywords: Solvency II; Technical Provisions; Binary Events; Events Not In Data; Binary Events Loading

CONTENTS

* Actuarial and Insurance Management Solutions, PwC UK, 7 More London Riverside, London SE1 2RT, UK // *WWW:* http://www.uk.pwc.com
Senior Manager, Risk and Capital // *E-mail:* yuriy.krvavych@uk.pwc.com

Under the Solvency II Directive[1] insurers are required to allow for all possible events when setting their technical provisions, including those that may not have been historically realised before. Such events not presented in a set of observable historical loss data are often called *Binary* events. Here, the term 'Binary' is traditionally used to define loss generating events with low frequency and high severity impact. An alternative name, *Events Not In Data (ENID)*, may also be used to denote a much broader set of unobservable loss events (e.g. see GIRO-2013 [5]). In its *Technical Provisions Guidance* issued in March 2011 (LLOYD'S [8]) whilst suggesting several approaches to allowing for Binary events, Lloyd's encourages actuarial functions to use one specific approach that offers uplifting reserve best estimates to allow for a limited range of historical (observable) data. This approach is also called the *Truncated Statistical Distribution* approach, and defines the **uplift factor** of reserve best estimate as:

"the ratio of the 'true mean' to the 'mean only including realistically foreseeable events' ",

using the following (subjective) assumptions

- the actuarial function can only observe loss events with a return period of up to Y years - in the Lloyd's guidance such events are called *'realistically foreseeable events'*;
- the true reserve distribution from which reserve values based on all observable and unobservable events are drawn has a certain parametric form F. Therefore, the reserve values based on the realistically foreseeable events are drawn from the truncated reserve distribution, or the distribution of conditional random reserve values given they are below a certain p-quantile defined by $F^{-1}(p)$, where $p = 1 - 1/Y$; and that
- the mean and variability of reserve based on the loss data only including realistically foreseeable events are available to the actuarial function.

Although the Lloyd's guidance does not provide any explicit analytical formulae for calculating the uplift factor, it seems the industry is in favour of the following two analytical approximations of the load of reserve mean, **Mean_Load**, assuming log-normality of the true reserve distribution (here **Mean_Load** is simply **uplift factor** minus 1):

$$\text{Lloyd's Formula 1:} \quad \boxed{\frac{p}{\Phi\left(\Phi^{-1}(p) - \sqrt{\ln\left(\mathrm{CoV_{tr}}^2 + 1\right)}\right)} - 1} \tag{1}$$

$$\text{Lloyd's Formula 2:} \quad \boxed{\frac{1}{\Phi\left(\Phi^{-1}(p) - \sqrt{\ln\left(\mathrm{CoV_{tr}}^2 + 1\right)}\right)} - 1} \tag{2}$$

where $\mathrm{CoV_{tr}}$ is the coefficient of variation of the reserve based on the truncated set of loss data representing realistically foreseeable events with the return period of up to $Y = 1/(1-p)$ years; Φ is the standard normal cumulative distribution function.

Both formulae, whilst having similar analytical structure, could produce a noticeably different result, as their corresponding **uplift factor** (or equivalently **Mean_Load** plus 1) differs by factor p. This implies that, since both formulae are analytical approximations, at least one of them will be a crude approximation.

1 See Solvency II Directive Article 77 (2) and also corresponding Groupe Consaltatif explanatory text (cf. Sandström [12]).

The purpose of this paper is to examine the quality of the two approximations (1) and (2) by comparing them to the exact (true) value of **Mean_Load**, and also provide analogous approximation of **Mean_Load** in a distribution-free setting assuming general (non-normal) characteristics of the reserve risk profile.

The structure of this paper is as follows. In section 2, the two Lloyd's approximations (1) and (2) are analytically derived from first principles. Here, the quality of the two approximations was analysed by comparing them to the exact (true) value of **Mean_Load** under the assumption of log-normality of reserve risk profile, and it was shown that (1) generally provides a much better approximation than its alternative (2). This section also provides some ideas of how the approximation (1) may be implemented in practice. In section 3, we focus on providing a practical approximation of ENID load for reserve mean, **Mean_Load**, in a distribution-free setting assuming general (non-normal) characteristics of the reserve risk profile. This section makes the following key assumptions used in the derivation of the ENID load approximation for reserve mean:

1. the shape of reserve distribution is completely defined (driven) by a single shape parameter;
2. for the centralised and normalised value of reserve its
   a) distribution is approximated by the Fleishman quadratic polynomial of a standard normal variable (see Fleishman [4]); and
   b) p-quantile is approximated using the Cornish-Fisher approximation of second order (see Fisher and Cornish [3]).

The assumption 1. allows to define any higher-order statistic, like skewness, kurtosis and other relative moments of higher order, as a function of unknown true value of CoV. The assumptions 2.a) and 2.b) use CoV and skewness only to explain the shape of the reserve risk profile once scaled by its mean. Altogether, this allows us to express the ENID load for reserve mean as a function of unknown true value of CoV. The unknown true value of CoV is ultimately solved for a given level of observed CoV_{tr} and further used to calculate the ENID load approximation. The obtained approximation results are tabulated by CoV_{tr}, p and the level of skewness per unit of CoV. Finally, brief conclusions are given in section 4.

2 DEMYSTIFYING LLOYD'S APPROXIMATIONS

2.1 *Derivation of approximations (1) and (2) from first principles*

We denote the reserve value based on all observable and non-observable loss events by random variable X that is log-normally distributed with unknown parameters μ and σ, i.e. $X \sim \mathcal{LN}(\mu, \sigma^2)$ and the cumulative distribution function of X is $F_X(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$. Here, it would be reasonable to assume that the actuarial function can observe loss events with a return period of at least 20 years, or equivalently $0.95 \leq p < 1$. The k-th non-central moment[2] of X is then equal to:

$$m^{(k)} = \mathbb{E}[X^k] = e^{\mu k + \frac{1}{2}\sigma^2 k^2}. \tag{3}$$

The reserve value based on realistically foreseeable (observable) loss events represent the subset of X, and is denoted by conditional random variable X_{obs} =

2 The k-th non-central moment of $\mathcal{LN}(\mu, \sigma^2)$ is the Moment Generating Function of $\mathcal{N}(\mu, \sigma^2)$ at point k, i.e. $mgf_{\mathcal{N}(\mu,\sigma)}(k) = \mathbb{E}\left[e^{k*\mathcal{N}(\mu,\sigma^2)}\right] = \mathbb{E}\left[e^{\mathcal{N}(k\mu,k^2\sigma^2)}\right]$.

$\{X|X \le b\}$, where $b = F_X^{-1}(p)$. The k-th non-central moment of X_{obs} is the k-th truncated non-central moment of X which is equal to:

$$
\begin{aligned}
m_{tr}^{(k)} = \mathbb{E}[X_{obs}^k] &= \mathbb{E}[X^k|X \le b] \\
&= m^{(k)} \frac{\Phi\left(\frac{\ln b - \mu}{\sigma} - k\sigma\right)}{\Phi\left(\frac{\ln b-\mu}{\sigma}\right)} \\
&= m^{(k)} \frac{\Phi\left(\Phi^{-1}(p) - k\sigma\right)}{p}.
\end{aligned}
\tag{4}
$$

The formula (4) of k-th truncated non-central moment of a log-normally distributed random variable is well covered in the research literature and can, for example, be found in Kotz et al. [7] (see p. 241 therein). The detailed derivation of (4) is provided in Appendix A.

The mean $m_{tr}^{(1)}$ and variability CoV_{tr} of reserve X_{obs} derived by the actuarial function using the set of observable loss events are then calculated as follows:

$$
\begin{aligned}
m_{tr}^{(1)} &= \mathbb{E}[X|X \le b] \\
&= m^{(1)} \frac{\Phi\left(\Phi^{-1}(p) - \sigma\right)}{p} \\
&= m^{(1)} \times \alpha,
\end{aligned}
\tag{5}
$$

where $\alpha = \frac{\Phi(\Phi^{-1}(p)-\sigma)}{p} \in (0,1)$ for $p \in [0.95, 1)$, and tends to 1 as p goes to 1; and

$$
\begin{aligned}
{\text{CoV}_{\text{tr}}}^2 &= \frac{m_{tr}^{(2)} - \left(m_{tr}^{(1)}\right)^2}{\left(m_{tr}^{(1)}\right)^2} = \frac{m_{tr}^{(2)}}{\left(m_{tr}^{(1)}\right)^2} - 1 \\
&\overset{(3)\&(4)}{=} \frac{\left(m^{(1)}\right)^2 e^{\sigma^2} \Phi\left(\Phi^{-1}(p) - 2\sigma\right)/p}{\left(m^{(1)}\right)^2 \alpha^2} - 1 \\
&= e^{\sigma^2} \frac{\Phi\left(\Phi^{-1}(p) - 2\sigma\right)}{p\,\alpha^2} - 1.
\end{aligned}
\tag{6}
$$

It follows from (5) that **Mean_Load** $= \frac{1}{\alpha} - 1$, and to find the exact value of unknown α one would need to solve the system of equations (5) and (6) by extracting $\sigma(\alpha) = \Phi^{-1}(p) - \Phi^{-1}(p\,\alpha)$ from (5), ploughing it into equation (6) and solving it for α numerically.

Alternatively, the right hand side of (6) can be approximated by $e^{\sigma^2} - 1$ for p values close to 1, as

- $\alpha(p) \to 1$, when $p \to 1$; and
- $\frac{\Phi(\Phi^{-1}(p)-2\sigma)}{p\,\alpha^2} \to 1$, when $p \to 1$.

Formally, it can be shown using the Asymptotic Analysis that

$$
\frac{\Phi\left(\Phi^{-1}(p) - 2\sigma\right)}{p\,\alpha^2} = 1 - \delta(p),
$$

where $0 < \delta(p) = 1 - \frac{\Phi(\Phi^{-1}(p)-2\sigma)}{p\,\alpha^2}$ and such that $\exists \theta_0 \in (0,1)$ for which $\forall \theta \in (0, \theta_0)$: $\delta(p) = o\left((1-p)^{\theta}\right)$. Here, the symbol o indicates a *little-o* function[3].

3 A real function f is called a *little-o* function of another real function g when $x \to x_0$, if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$, or in other words f tends to 0 faster than g. We denote $f(x) = o(g(x))$, $(x \to x_0)$. For formal definition of little-o functions please refer to de Bruijn [2].

From here we get the estimate $\widehat{\sigma} = \sqrt{\ln\left(\text{CoV}_{\text{tr}}{}^2 + 1\right)}$, and use it further to approximate **Mean_Load**

$$
\begin{aligned}
\widehat{\textbf{Mean\_Load}} &= \frac{1}{\alpha} - 1, \\
&= \frac{p}{\Phi\left(\Phi^{-1}(p) - \widehat{\sigma}\right)} - 1, \\
&= \frac{p}{\Phi\left(\Phi^{-1}(p) - \sqrt{\ln\left(\text{CoV}_{\text{tr}}{}^2 + 1\right)}\right)} - 1.
\end{aligned}
$$

Further approximation is then obtained from $\widehat{\textbf{Mean\_Load}}$ by assuming $p \approx 1$, in which case we have a crude approximation

$$
\widehat{\widehat{\textbf{Mean\_Load}}} = \frac{1}{\Phi\left(\Phi^{-1}(p) - \sqrt{\ln\left(\text{CoV}_{\text{tr}}{}^2 + 1\right)}\right)} - 1.
$$

2.2 *Quality of Lloyd's approximations*

It follows from above that the first Lloyd's formula (1) represents a fine approximation of **Mean_Load**, whereas the second formula (2) is a crude approximation. The quality of those two approximations obviously depends on the value of observed variability of reserve, CoV_{tr}, and the probability truncation point p - i.e., the smaller the value of CoV_{tr} and the closer the value of p to 1, the closer approximation is to the true value of **Mean_Load**. This is shown in Figure 1. The dynamic visualisation of the functional relationship between **Mean_Load** and p assuming varying value of CoV_{tr} is presented in Figure 2 in the form of *animated graph*.

Figure 1: *The quality of Lloyd's approximation formulae under different values of* CoV_{tr}.

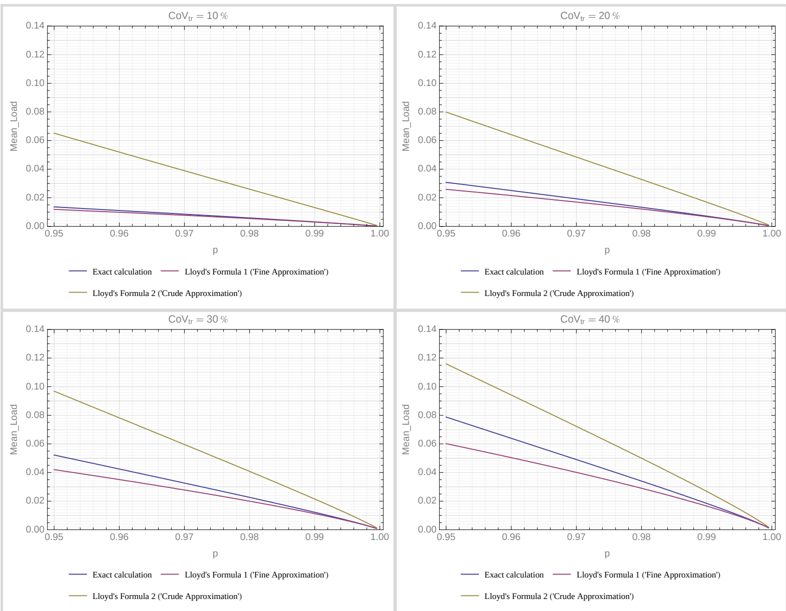
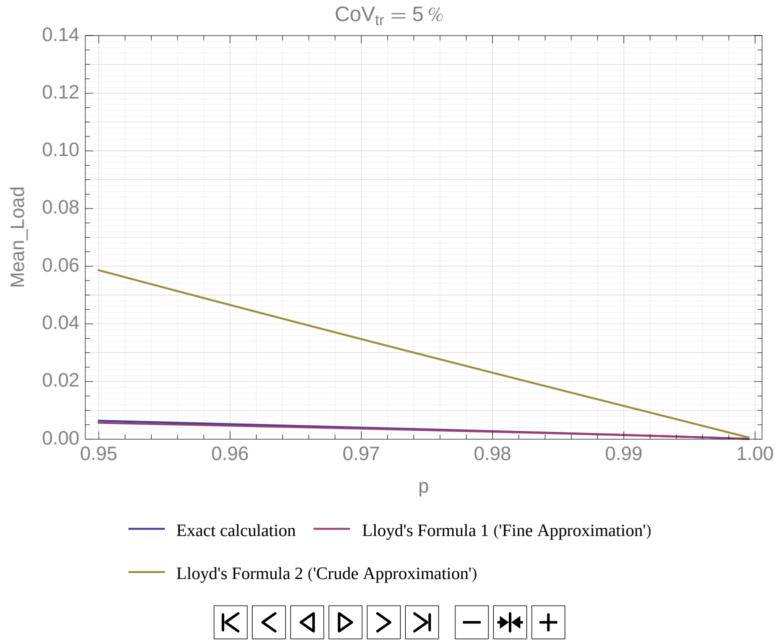


Figure 2: *Animated graph of functional relationship between* **Mean_Load** *and p under varying value of* $\mathrm{CoV}_{\mathrm{tr}}$. *Please use the control buttons to play the animation.*



2.3 *Practical implementation of Lloyd's approximation (1)*

In an ideal world one could use a numerical solver (e.g. using VB, Matlab, R or Mathematica) to compute the exact value of **Mean_Load** for a given pair of values (p, $\mathrm{CoV}_{\mathrm{tr}}$). In reality, for a reserving actuary with limited ability to use software for numerical computations, it might be more practical to implement the Lloyd's formula (1) in MS Excel with the help of additional correction factors k that would bring the approximation infinitesimally close to the true exact value. The implemented value of **Mean_Load** is then calculated as

$$\widetilde{\mathbf{Mean\_Load}} = k \times \widehat{\mathbf{Mean\_Load}}.$$

The Table 1 below presents a grid of pre-computed correction factors defined as the ratio of exact value to approximated value of **Mean_Load** calculated using formula (1).

Table 1: *Correction factors for the Lloyd's formula (1).*

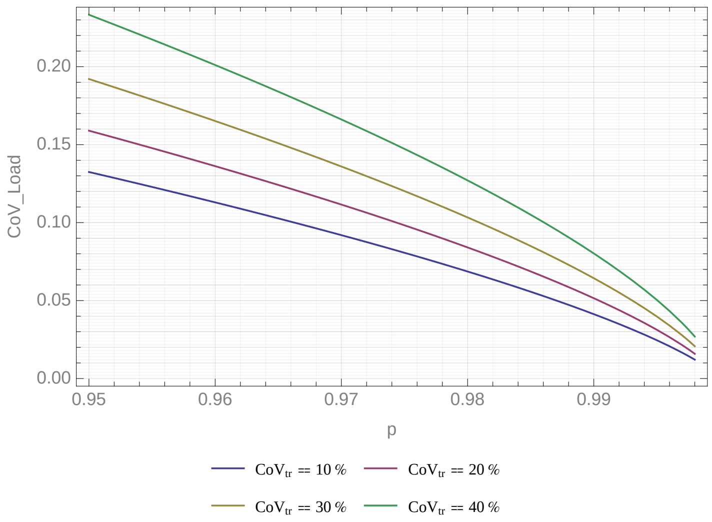
	p = 0.95	**p = 0.955**	**p = 0.96**	**p = 0.965**	**p = 0.97**	**p = 0.975**	**p = 0.98**	**p = 0.985**	**p = 0.99**
$\mathrm{CoV}_{\mathrm{tr}}$ =									
5%	1.127	1.118	1.109	1.099	1.088	1.077	1.066	1.053	1.039
10%	1.146	1.135	1.125	1.113	1.102	1.089	1.076	1.062	1.046
15%	1.166	1.155	1.142	1.130	1.117	1.103	1.088	1.072	1.054
20%	1.189	1.176	1.163	1.149	1.134	1.118	1.102	1.084	1.063
25%	1.215	1.200	1.185	1.169	1.153	1.136	1.117	1.097	1.074
30%	1.243	1.227	1.210	1.193	1.174	1.155	1.134	1.111	1.085
35%	1.275	1.257	1.238	1.218	1.198	1.176	1.152	1.127	1.098
40%	1.309	1.289	1.268	1.246	1.223	1.199	1.173	1.145	1.112
45%	1.348	1.325	1.301	1.277	1.252	1.225	1.196	1.164	1.128
50%	1.389	1.364	1.338	1.311	1.282	1.252	1.220	1.185	1.145

Here, for example, for a particular reserve class with the historical data including observable loss events with the return period of up to 50 years, i.e. $p = 0.98$, and the estimated reserve variability $\mathrm{CoV}_{\mathrm{tr}} = 18.5\%$, the corresponding correction coefficient is determined to be 1.102 (highlighted in grey) via using an interpolated grid search.

Although the Solvency II Directive does not require uplifting the variability assumption when allowing for the events not in data, one could take a step further and calculate the CoV load using the following algorithm:

1) $\widetilde{\alpha} = \frac{1}{1+\widetilde{\textbf{Mean\_Load}}}$;
2) $\widetilde{\sigma} = \sigma(\widetilde{\alpha}) = \Phi^{-1}(p) - \Phi^{-1}(p\,\widetilde{\alpha})$;
3) $\widetilde{\text{CoV}}^2 = e^{\widetilde{\sigma}^2} - 1$;
4) calculate the reserve CoV load: $\textbf{CoV\_Load} = \frac{\widetilde{\text{CoV}}}{\text{CoV}_{\text{tr}}} - 1$.

The graph below provides the reserve CoV load as a function of p under different values of CoV_{tr}.



3 ENID LOAD APPROXIMATIONS IN A DISTRIBUTION-FREE SETTING

The Lloyd's approximation formulae in section 2 were derived under the assumption that the true distribution of reserve is a Log-Normal distribution $\mathcal{LN}(\mu, \sigma^2)$. There, the ENID load for reserve mean is defined by the distribution's shape parameter σ and the probability truncation point p, and is invariant with respect to the distribution's scale parameter μ. This is mainly because of the special features of Log-Normal distribution. In particular, it is the kind of distribution for which its scale and shape parameters are clearly separated, implying that the shape of the distribution, that is in general driven by skewness, kurtosis and other relative moments of higher order[4], is defined only by the distribution's shape parameter.

In the case of Log-Normal distribution, its level of variability CoV and its shape parameter σ are directly related to each other:

$$\text{CoV}^2 = e^{\sigma^2} - 1,$$

and hence its skewness and all other relative moments of higher order are purely defined by CoV. In particular, the skewness of Log-Normal distribution is characterised by so called *Skewness-to-CoV (SC) ratio*, which, being $SC = e^{\sigma^2} + 2 = \text{CoV}^2 + 3$, falls into the range of $(3, 4)$ for realistic values of CoV < 100%.

4 As per asymptotic expansion of an analytical CDF using the Cornish-Fisher expansion series (see Fisher and Cornish [3]).

Whilst being a commonly used distribution in insurance, the Log-Normal distribution is still not suitable for modelling the whole range of practically feasible reserve risk profiles, and specifically those with relatively lower or higher skewness, i.e. when respectively $SC < 3$ or $SC > 4$.

This section focuses on providing a distribution-free approach to estimating ENID load for reserve mean and variability. The proposed approach utilises the idea of estimating reserve risk profile, once normalised by its mean (location), by the level of its variability, CoV, and its skewness which is assumed to be at a constant level of SC once measured per unit of CoV. The structure of this section is as follows. In subsection 3.1, we define the range of practically feasible reserve risk profiles. Here, reserve risk profiles are characterised by the type of business or reserving class, e.g. personal vs. commercial insurance, short vs. long tail, duration and convexity of reserve claims payments, etc. This characterisation in turn implies the differentiation of reserve distributions by the level of variability, CoV, and the level of skewness per unit of CoV, i.e. SC ratio. In subsection 3.2, we develope a distribution-free approach to estimating ENID load for reserve mean and variability. This utilises the following two known distribution approximations - the *Fleishman approximation* and the *Cornish-Fisher approximation*. Both are used to approximate the reserve distribution using its CoV and skewness only. The obtained approximations are derived using both analytical transformations and numerical calculations, and tabulated for different levels of CoV_{tr}, probability truncation point p and SC ratio.

3.1 *Reserve risk profile: differentiation by type of business class*

In practice, non-life reserving actuaries often use *Coefficient of Variation* (CoV) as the measure of riskiness of modelled reserves. For example, personal lines like motor and home are short tail business lines and exhibit relatively lower CoV when compared to long tail classes like commercial liability. However, CoV alone cannot explain all the characteristics of the reserve risk profile, and thus higher moments of reserve distribution like skewness and kurtosis would be required to properly capture a) the degree of asymmetry of odds towards adverse reserve realisations, and b) heavy-tailedness of the reserve distribution.

In general, the parametric distributions commonly used in insurance for reserving and loss modelling are of a special type:

- they are often defined by two parameters - the *scale parameter* and the *shape parameter*; and
- their shape is totally driven by a single parameter - their shape parameter.

Equivalently, those two-parameter distributions are such, that when scaled by their mean (location), would have a unique fixed location (unit mean) and variable shape dependent on the shape parameter only, i.e. the distribution of the following random variable Y is a single-parameter distribution and its shape is defined by the shape parameter of X:

$$Y = \frac{1}{m_X} X = 1 + \text{CoV}_X \cdot \widetilde{X}, \tag{7}$$

where $\widetilde{X} = \frac{X - m_X}{m_X \, \text{CoV}_x}$ and m_X and CoV_X are the mean and CoV of X respectively.

In this paper, we focus only on the class of two-parameter distributions with *single shape parameter* and denote it by $\mathcal{SSP}$. Examples of two-parameter distributions of $\mathcal{SSP}$ type include[5] Gamma, Inverse-Gaussian (Wild), Log-Normal, Dagum, Suzuki, Exponentiated-Exponential (Verhulst), Inverse-Gamma (Vinci), Birnbaum-Saunders, Exponentiated-Fréchet and Log-Logistic. It should be

5 Please refer to Kleiber and Kotz [6], Nadarajah and Kotz [11], Nadarajah [10], Marshall and Olkin [9] and Wolfram Documentation Center [14].

noted, that not all two-parameter distributions are of $\mathcal{SSP}$ type, and immediate example of that would be the Log-Gamma distribution each of the two parameters of which would drive both the location/scale and the shape of the distribution at the same time.

As was illustrated above in (7), for any distribution of $\mathcal{SSP}$ type its shape in general and its CoV in particular are defined by the distribution's shape parameter only. Also, it is known fact from the Distribution Analysis of the Probability Theory that any analytical cumulative distribution function can be expanded using the Cornish-Fisher expansion (cf. Fisher and Cornish [3]), which utilises the distribution's skewness, kurtosis[6] and other relative moments of higher order to fully explain its shape. In the case of random variable $\widetilde{X}$, the shape of its distribution is completely explained by the skewness, kurtosis and other relative moments of higher order of X. This implies that all relative moments of third order and higher of any distribution of $\mathcal{SSP}$ type are completely defined by its shape parameter.

Therefore, the distinctive features of the distributions of $\mathcal{SSP}$ type are:

- their shape parameter is a function of CoV;
- their any higher-order statistic is fully determined by the shape parameter and hence is a function of CoV, and in particular
  - relative skewness measured by Skewness-to-CoV (SC) ratio is a function of CoV;
  - relative kurtosis measured by Kurtosis-to-CoV^2 (KCsq) ratio is a function of CoV^2;
- if the $\mathcal{SSP}$ distribution belongs to a certain parametric family (e.g. Log-Normal, Gamma or any other parametric family from $\mathcal{SSP}$) then scaling it by its mean preserves the parametric family it belongs to and its shape parameter, i.e. if $X \sim F_{u,v}$ with mean m_X and standard deviation s_X, scale parameter u and shape parameter v, then $Y = \frac{1}{m_X} X \sim F_{\widetilde{u},v}$ with mean 1 and standard deviation CoV_X, and the scale parameter $\widetilde{u}$ is a function of shape parameter v.

The latter feature also implies that

$$F_{u,v}(x) = F_{\widetilde{u},v}\left(\frac{x}{m_X}\right). \tag{8}$$

The $\mathcal{SSP}$ distributions can be split into three main categories:

- **Moderately skewed distributions** ($1.5 < SC \leq 3$)
  - *Gamma*;
  - *Inverse-Gaussian (Wald)*;
- **Significantly skewed distributions** ($3 < SC < 4$)
  - *Log-Normal*;
  - *Suzuki*;
  - *Exponentiated-Exponential (Verhulst)*;
  - *Dagum*;
- **Extremely skewed distributions** ($4 < SC < 5.5$)
  - *Inverse-Gamma (Vinci)*;
  - *Birnbaum-Saunders*;

6 Here, kurtosis is regarded as excess-kurtosis and thus is defined via the fourth- and second-order cumulants of the reserve distribution.

– *Log-Logistic*;
– *Exponentiated-Fréchet*.

Assumption 1. This paper considers the following four parametric distributions of $\mathcal{SSP}$ type that in total cover a wide enough range of practically feasible reserve risk profiles, i.e. for $1.5 < SC < 5.5$:

- *Gamma*;
- *Inverse-Gaussian (Wald)*;
- *Log-Normal*; and
- *Inverse-Gamma (Vinci)*,

and assumes that any reserve risk profile with $SC = k_S \times \mathrm{CoV}$ for a given level of CoV (k_S is a positive multiplier) could be associated with the closest distribution curve out of the four parametric distributions considered. We denote the set of the four proposed parametric distributions by $\mathcal{SSP}' \subset \mathcal{SSP}$. This is the key assumption of reserve risk profile characterisation that is further used in the derivation of ENID load approximation formulae in subsection 3.2.

We further use the $\mathcal{SSP}'$ set of parametric distributions to illustrate how reserve risk profiles could be differentiated by type of business/reserving class. This is presented in Table 2.

Table 2: *Differentiation of reserve risk profile by type of reserve class.*

Type of reserving class				
Duration	**CoV range**	**Skewness (SC ratio)**	**Parametric distribution(s)**	**Example of reserving class**
Short tail	10%-12%	1.9 to 2.1	Gamma	Motor (ex Bodily Injury)
Short tail	12%-16%	2.0 to 3.0	Gamma, Inverse-Gaussian (Wald)	Home
Short tail	10%-16%	2.9 to 3.1	Inverse-Gaussian (Wald), Log-Normal	Comm Property/Fire, Comm Accident
Long tail	12%-25%	3.0 to 3.5	Log-Normal	Motor Bodily Injury, Marine
Long tail	18%-50%	3.0 to 4.0	Log-Normal, Inverse-Gamma (Vinci)	Workers Comp, Prof Liab, Comm Liab
Long tail	25%-70%	> 4	Inverse-Gamma (Vinci)	Asbestos and other long tail books

The characteristics of the four proposed parametric distributions in $\mathcal{SSP}'$ are provided in Table 3 below. There, $KCsq$ is *Kurtosis-to-CoV^2* ratio, which measures kurtosis per unit of CoV^2.

Table 3: *SC and KCsq ratios for the four parametric distributions.*

Parametric distribution	**SC ratio as a function of CoV**	**KCsq ratio as a function of CoV^2**
Gamma	2	6
Inverse-Gaussian (Wald)	3	15
Log-Normal	$3 + \mathrm{CoV}^2 \in (3, 4),\ \mathrm{CoV} < 100\%$	$16 + 15\mathrm{CoV}^2 + 6\mathrm{CoV}^4 + \mathrm{CoV}^6 > 16$
Inverse-Gamma (Vinci)	$\frac{4}{1-\mathrm{CoV}^2} > 4,\ \mathrm{CoV} < 100\%$	$\frac{30\left(1-\frac{1}{5}\mathrm{CoV}^2\right)}{\left(1-\mathrm{CoV}^2\right)\left(1-2\mathrm{CoV}^2\right)} > 30,\ \mathrm{CoV} < 70\%$

The graphs of Skewness and Kurtosis as functions of CoV derived from SC and $KCsq$ ratios are provided for $\mathcal{SSP}'$ set of parametric distributions below in Figure 3 and Figure 4.

Figure 3: *Skewness as a function of* CoV *for the four parametric distributions from* $\mathcal{SSP}'$.

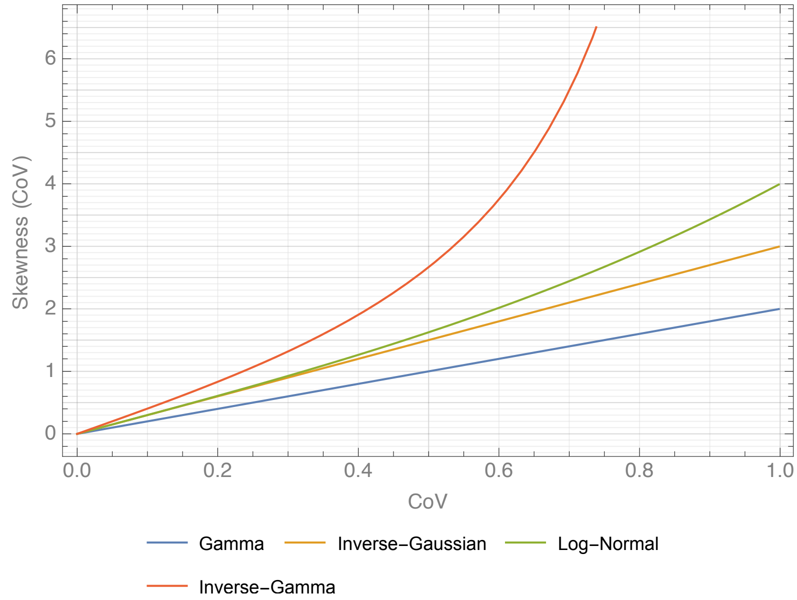
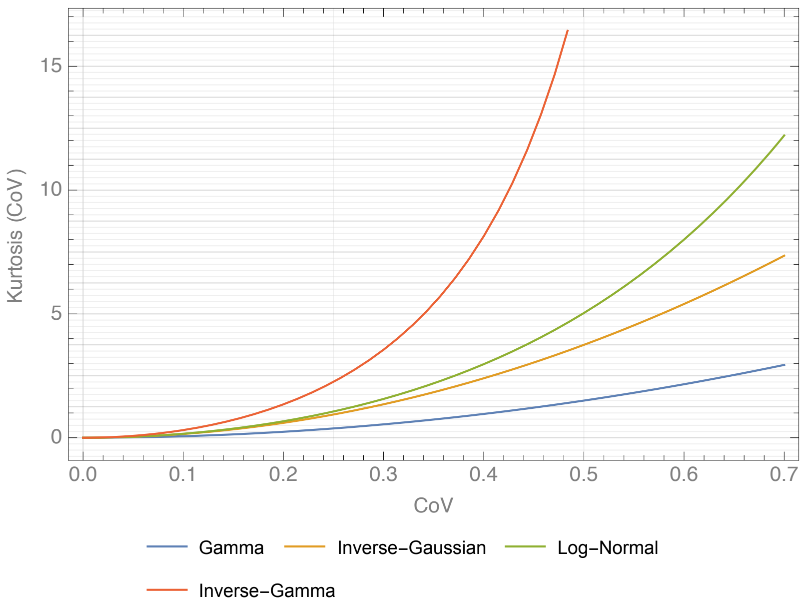


Figure 4: *Kurtosis as a function of* CoV *for the four parametric distributions from* $\mathcal{SSP}'$.



These graphs demonstrate monotonic increase in the level of Skewness and Kurtosis for a given level of CoV when moving sequentially across the $\mathcal{SSP}'$ set of the proposed four parametric distributions from Gamma to Inverse-Gamma.

ASSUMPTIONS AND ANALYTICAL TRANSFORMATIONS We consider the random value of reserve X with its CDF F_X and unknown true values of mean m, standard deviation s and coefficient of variation $\text{CoV} = \frac{s}{m}$. As was defined in section 2 the conditional random variable $X_{obs} = \{X \mid X \le b\}$, where $b = F_X^{-1}(p) = \text{VaR}_p(X)$, represents the reserve value based on realistically foreseeable (observable) loss events. The n-th non-central moment of X_{obs} is the n-th truncated non-central moment of X which is equal to:

$$m_{tr}^{(n)}(X) = \mathbb{E}\left[X_{obs}^n\right] = \mathbb{E}[X^n \mid X \le b]. \tag{9}$$

Using the representation (7) we rewrite X as

$$X = m\left(1 + \text{CoV} \cdot \widetilde{X}\right),$$

where $\widetilde{X} = \frac{X-m}{s}$, and note that since

$$\mathbb{P}\left[\widetilde{X} \le \text{VaR}_p\left(\widetilde{X}\right)\right] = \mathbb{P}\left[X \le \text{VaR}_p\left(X\right)\right] = p,$$

we have

$$\text{VaR}_p\left(X\right) = m + s \cdot \text{VaR}_p\left(\widetilde{X}\right) \tag{10}$$

and

$$\begin{aligned} m_{tr}^{(n)}(X) &= \mathbb{E}\left[X^n \mid X \le \text{VaR}_\alpha(X)\right] \\ &= m^n \cdot \mathbb{E}\left[Y^n \mid Y \le \text{VaR}_\alpha(Y)\right] \\ &= m^n \cdot m_{tr}^{(n)}(Y), \end{aligned} \tag{11}$$

where $Y = \frac{1}{m} \cdot X = 1 + \text{CoV} \cdot \widetilde{X}$. The above formulae of linearity of VaR transformation (10) and the factorisation of n-th truncated moment (11) hold for any type of distribution of X. This is used further to calculate the truncated mean and variance of X.

The truncated mean of X is calculated as follows

$$\begin{aligned} m_{tr} = m_{tr}^{(1)}(X) &= \mathbb{E}[X \mid X \le \text{VaR}_p\left(X\right)] \\ &= m\,\mathbb{E}\left[\left(1 + \text{CoV} \cdot \widetilde{X}\right) \mid \widetilde{X} \le \text{VaR}_p\left(\widetilde{X}\right)\right] \\ &= m\left(1 + \text{CoV} \cdot \mathbb{E}\left[\widetilde{X} \mid \widetilde{X} \le \text{VaR}_p\left(\widetilde{X}\right)\right]\right), \end{aligned} \tag{12}$$

from where the ENID uplift factor[7] for reserve mean is defined as

$$\boxed{\frac{m}{m_{tr}} = \frac{1}{1 + \text{CoV} \cdot \mathbb{E}\left[\widetilde{X} \mid \widetilde{X} \le \text{VaR}_p\left(\widetilde{X}\right)\right]}} \tag{13}$$

The truncated variance of X is calculated as follows

$$\begin{aligned} s_{tr}^2 &= \mathbb{E}\left[\left(X - m_{tr}\right)^2 \mid X \le \text{VaR}_p\left(X\right)\right] \\ &= m^2\,\text{CoV}^2\,\mathbb{E}\left[\left(\widetilde{X} - \mathbb{E}\left[\widetilde{X} \mid \widetilde{X} \le \widetilde{b}\right]\right)^2 \mid \widetilde{X} \le \widetilde{b}\right] \\ &= m^2\,\text{CoV}^2\left(\mathbb{E}\left[\widetilde{X}^2 \mid \widetilde{X} \le \widetilde{b}\right] - \mathbb{E}^2\left[\widetilde{X} \mid \widetilde{X} \le \widetilde{b}\right]\right), \end{aligned} \tag{14}$$

where $\widetilde{b} = \text{VaR}_p\left(\widetilde{X}\right)$. By multiplying both sides of Equation 14 by squared $\frac{m}{m_{tr}}$ from Equation 13 and then dividing them by m^2, we obtain the following formula for CoV_{tr}^2:

$$\boxed{\text{CoV}_{tr}^2 = \frac{\text{CoV}^2 \cdot \left(\mathbb{E}\left[\widetilde{X}^2 \mid \widetilde{X} \le \widetilde{b}\right] - \mathbb{E}^2\left[\widetilde{X} \mid \widetilde{X} \le \widetilde{b}\right]\right)}{\left(1 + \text{CoV} \cdot \mathbb{E}\left[\widetilde{X} \mid \widetilde{X} \le \widetilde{b}\right]\right)^2}} \tag{15}$$

7 The ENID load is then the ENID uplift factor minus 1.

It follows from Equation 13 and Equation 15 that the ENID load for reserve mean and the truncated coefficient of variation CoV^2_{tr} are functionally dependent on CoV and the first two truncated moments of $\widetilde{X}$. To be able to proceed further and approximate ENID load for reserve mean and variability, we will first try to approximate the truncated moments of $\widetilde{X}$ - the centralised and normalised random variable with zero mean and unit variance, by a function of CoV, and then solve Equation 15 for CoV and use it to calculate $\frac{m}{m_{tr}}$ in Equation 13 and also $\frac{\mathrm{CoV}}{\mathrm{CoV}_{tr}}$. To be able to achieve this we will make some simplifications and series of analytical transformations outlined below.

Assumption 2. The random variable $\widetilde{X}$ is assumed to be approximated by the Fleishman quadratic polynomial of a standard normal random variable $Z \sim \mathcal{N}(0,1)$:

$$\widetilde{X} \overset{d}{\approx} P_2(Z) = a_1 Z + a_2(Z^2 - 1), \tag{16}$$

where the Fleishman coefficients a_1 and a_2 are calibrated so that $P_2(Z)$ has unit variance and its skewness is equal to γ - skewness of X, i.e. a_1 and a_2 are solution to the following system of equations

$$\begin{cases} 1 &= a_1^2 + 2a_2^2, \\ \gamma &= 6a_1^2 a_2 + 8a_2^3 \end{cases} \tag{17}$$

Assumption 3. The p-quantile of random variable $\widetilde{X}$ is assumed to be approximated by the Cornish-Fisher approximation:

$$\widetilde{b} = \mathrm{VaR}_p\left(\widetilde{X}\right) \approx z_p + \gamma\,\frac{z_p^2 - 1}{6}, \tag{18}$$

where $z_p = \mathrm{VaR}_p\,(Z)$.

Assumption 4. The empirical distribution of reserve in a distribution-free setting is assumed to be of $\mathcal{SSP}$ type, and thus the unknown true value of skewness is defined by

$$\gamma\,(\mathrm{CoV}) = SC\,(\mathrm{CoV}) \cdot \mathrm{CoV}. \tag{19}$$

It is further assumed that the SC ratio is constant in the neighbourhood of CoV_{tr}, i.e. for CoV in the interval $(\mathrm{CoV}_{tr}, (1+\delta)\mathrm{CoV}_{tr})$ for $0 < \delta \ll 1$:

$$SC\,(\mathrm{CoV}) = k, \tag{20}$$

and the multiple k is thus commensurate with the SC ratio on the truncated basis, i.e.

$$k \approx \frac{\gamma_{tr}}{\mathrm{CoV}_{tr}}. \tag{21}$$

Furthermore we make the following analytical transformations:

- Calibrate Fleishman coefficients a_1 and a_2, and express them as functions of CoV;
- Express the n-th truncated moment $\mathbb{E}\left[\widetilde{X}^n \,|\, \widetilde{X} \le \widetilde{b}\right]$ for $n = 1, 2$ as a function of CoV by:
  - finding the equivalent probability condition of $\left\{\widetilde{X} \le \widetilde{b}\right\}$ defined through the random variable Z – this would likely to be in the following form $\{c \le Z \le d\}$ with c and d being functions of CoV; and then

– calculating the n-th truncated moment of the standard normal random variable Z, i.e. $\mathbb{E}\left[Z^n \,|\, c \le Z \le d\right]$ for $n = 1, \ldots, 4$.

The calibration of Fleishman coefficients requires solving a cubic equation. Indeed, (17) can be reduced to the following system of equations

$$\begin{cases} a_1 &= \sqrt{1 - 2a_2^2}, \\ \gamma &= 6a_2 - 4a_2^3 \end{cases} \tag{22}$$

From (22) it follows that $|a_2| \le \frac{1}{\sqrt{2}}$ and thus $0 \le \gamma \le 2\sqrt{2}$, which may indicate that the Fleishman approximation is not suitable for the types of reserve risk profile that are adhering to Inverse-Gamma parametric distribution and have CoV above 50%. The discriminant of the cubic equation in (22) is equal to $\frac{\gamma^2}{64} - \frac{1}{8}$ and is negative for $\gamma < 2\sqrt{2}$, indicating that there are three real roots. It can be shown that after taking into account the lower and upper bounds of a_2 there is only one real root, which is positive and equal to:

$$a_2\left(\text{CoV}\right) = \sqrt{2}\cos\left(\frac{\varphi(\text{CoV})}{3} + \frac{4\pi}{3}\right), \tag{23}$$

where $\varphi\left(\text{CoV}\right) = \arccos\left(-\frac{\gamma(\text{CoV})}{2\sqrt{2}}\right)$. The derivation of the roots of cubic equation in (22) is provided in Appendix B.

It then follows from (22) and (23) that

$$a_1\left(\text{CoV}\right) = \sqrt{1 - 2a_2^2\left(\text{CoV}\right)} \tag{24}$$

To find the equivalent probability condition of $\left\{\widetilde{X} \le \widetilde{b}\right\}$ defined through the random variable Z, we would need to solve the following quadratic inequality for Z:

$$a_2 Z^2 + a_1 Z - \left(a_2 + \widetilde{b}\right) \le 0, \tag{25}$$

where $\widetilde{b}$ is defined in Equation 18 and is a positive function of CoV for $p > 0.85$ and $\forall\ \gamma > 0$. Given $a_2 > 0$, the solution to quadratic inequality (25) is the interval $c \le Z \le d$, where

$$c = \frac{-a_1 - \sqrt{a_1^2 + 4a_2\left(a_2 + \widetilde{b}\right)}}{2a_2}, \tag{26}$$

$$d = \frac{-a_1 + \sqrt{a_1^2 + 4a_2\left(a_2 + \widetilde{b}\right)}}{2a_2}. \tag{27}$$

It should be noted that $c < 0 < d$ as $a_1 > 0$.

Let us further denote the n-th truncated moment of Z by

$$I_n = \mathbb{E}\left[Z^n \,|\, c \le Z \le d\right], \; n \ge 0. \tag{28}$$

Then using the Fleishman approximation of $\widetilde{X}$ (16) and also (26) and (27), we obtain

$$\mathbb{E}\left[\widetilde{X} \,|\, \widetilde{X} \le \widetilde{b}\right] = a_2 I_2 + a_1 I_1 - a_2 I_0, \tag{29}$$

$$\begin{aligned} \mathbb{E}\left[\widetilde{X}^2 \,|\, \widetilde{X} \le \widetilde{b}\right] &= a_2^2 I_4 + 2a_1 a_2 I_3 + \left(1 - 4a_2^2\right) I_2 \\ &\quad -2a_1 a_2 I_1 + a_2^2 I_0. \end{aligned} \tag{30}$$

The n-th truncated moment of Z in (28) can be computed iteratively using the following formula

$$\begin{cases} I_n &= -\frac{d^{n-1}\varphi(d) - c^{n-1}\varphi(c)}{\Phi(d) - \Phi(c)} + (n-1) I_{n-2}, \text{ with} \\ I_1 &= -\frac{\varphi(d) - \varphi(c)}{\Phi(d) - \Phi(c)}, \text{ and} \\ I_0 &= 1. \end{cases} \tag{31}$$

The derivation of formulae (31) is provided in Appendix C. It then follows from (31) that

$$\begin{cases} I_2 &= -\frac{d\varphi(d)-c\varphi(c)}{\Phi(d)-\Phi(c)} + 1, \\ I_3 &= -\frac{d^2\varphi(d)-c^2\varphi(c)}{\Phi(d)-\Phi(c)} + 2I_1, \\ I_4 &= -\frac{d^3\varphi(d)-c^3\varphi(c)}{\Phi(d)-\Phi(c)} + 3I_2. \end{cases} \quad (32)$$

By summarising all the steps above we conclude that both $\mathbb{E}\left[\widetilde{X} \,|\, \widetilde{X} \le \widetilde{b}\right]$ and $\mathbb{E}\left[\widetilde{X}^2 \,|\, \widetilde{X} \le \widetilde{b}\right]$ are analytical functions of CoV, as

- I_n is a function of c and d (see (31));
- both c and d are functions of a_1, a_2 and $\widetilde{b}$ (see (26) and (27)); and finally
- a_1, a_2 and $\widetilde{b}$ are analytical functions of $\gamma(\mathrm{CoV})$ (see (24), (23), (18) and (19)).

PRACTICAL APPROXIMATIONS We use numerical computations to solve[8] Equation 15 for unknown true value of CoV. This solution is further used to compute

- **Mean_Load** $= \frac{m}{m_{tr}} - 1$ by using Equation 13; and
- **CoV_Load** $= \frac{\mathrm{CoV}}{\mathrm{CoV}_{\mathrm{tr}}} - 1$.

To examine the quality of the obtained ENID load approximations one would ideally need to compare them to analogous exact values of ENID load for each of the parametric distribution from $\mathcal{SSP}'$. In this paper, we examine the quality of the proposed distribution-free approximation of **Mean_Load** and **CoV_Load** for the following three parametric distributions from $\mathcal{SSP}'$: Log-Normal, Gamma and Inverse-Gamma.

It follows from (11) that in general for any random variable X its truncated CoV is computed as follows

$$\mathrm{CoV}_{\mathrm{tr}}{}^2(X) = \frac{m_{tr}^{(2)}(Y)}{\left(m_{tr}^{(1)}(Y)\right)^2} - 1, \quad (33)$$

where $Y = \frac{1}{m}X$. And if additionally the distribution of X is of SSP type and its CDF is $F_{u,v}$ with u and v being scale and shape parameters respectively, then, as was explained by Equation 8 in subsection 3.1, Y has the distribution $F_{\widetilde{u},v}(x)$ for which $\widetilde{u}$ and v are such that the mean of Y is 1, and thus the scale parameter $\widetilde{u}$ is a function of shape parameter v.

The unknown shape parameter v can then be calibrated by solving Equation 33 and further used to find the unknown true value of CoV of X:

$$\mathrm{CoV}(X) = \sqrt{\frac{m^{(2)}(Y)}{\left(m^{(1)}(Y)\right)^2} - 1}, \quad (34)$$

where $m^{(n)}(Y)$ is the n-th non-central (full) moment of Y.

Specifically, for the three chosen distributions from $\mathcal{SSP}'$ we have:

- *Log-Normal* distribution
  - $X \sim F_{\mu,\sigma}(x) = \mathcal{LN}(\mu,\sigma) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$ and $\widetilde{\mu}(\sigma) = -\frac{1}{2}\sigma^2$;
  - $\mathrm{CoV}_{\mathrm{tr}}{}^2(X) = pe^{\sigma^2}\frac{\Phi(\Phi^{-1}(p)-2\sigma)}{\Phi^2(\Phi^{-1}(p)-\sigma)} - 1$ and $\mathrm{CoV} = \sqrt{e^{\sigma^2} - 1}$;

8 The Brent's method (see Teukolsky et al. [13]) was used to solve Equation 15.

- *Gamma* distribution
  - $X \sim F_{\beta,\alpha}(x) = \text{Gamma}(\alpha,\beta) = \frac{\Gamma(\alpha)-\Gamma\left(\alpha,\frac{x}{\beta}\right)}{\Gamma(\alpha)}$ and $\widetilde{\beta}(\alpha) = \frac{1}{\alpha}$;
  - $\text{CoV}_{\text{tr}}{}^2(X) = \frac{\Gamma(2+\alpha)-\Gamma(2+\alpha,\Upsilon(\alpha,p))}{(\Gamma(1+\alpha)-\Gamma(1+\alpha,\Upsilon(\alpha,p)))^2}\left(\Gamma(\alpha)-\Gamma(\alpha,\Upsilon(\alpha,p))\right)$, where

$$\begin{aligned}\Gamma(\alpha) &= \int_0^\infty t^{\alpha-1}e^{-t}\,dt;\\ \Gamma(\alpha,s) &= \int_s^\infty t^{\alpha-1}e^{-t}\,dt;\\ \Upsilon(\alpha,p) &= \text{the solution for } z \text{ in } p = \frac{\Gamma(\alpha,z)}{\Gamma(\alpha)};\end{aligned}$$

  - $\text{CoV} = \frac{1}{\sqrt{\alpha}}$;
- *Inverse-Gamma* distribution
  - $X \sim F_{\beta,\alpha}(x) = \text{InvGamma}(\alpha,\beta) = \frac{\Gamma\left(\alpha,\frac{\beta}{x}\right)}{\Gamma(\alpha)}$ and $\widetilde{\beta}(\alpha) = \alpha - 1$;
  - $\text{CoV}_{\text{tr}}{}^2(X) = \frac{\Gamma(2+\alpha,\overline{\Upsilon}(\alpha,p))}{\Gamma^2(1+\alpha,\overline{\Upsilon}(\alpha,p))}\Gamma(\alpha,\overline{\Upsilon}(\alpha,p))$, where

$$\overline{\Upsilon}(\alpha,p) = \text{the solution for } z \text{ in } p = \frac{\Gamma(\alpha)-\Gamma(\alpha,z)}{\Gamma(\alpha)};$$

  - $\text{CoV} = \frac{1}{\sqrt{\alpha-2}}$.

For each of the three parametric distributions from $\mathcal{SSP}'$ we simultaneously run the following two procedures of estimating ENID load

1) approximating **Mean_Load** and **CoV_Load** by using the distribution-free approach defined above in this section; and
2) numerically solving[9] Equation 33 for unknown shape parameter by using the above analytical formulae, then finding the unknown CoV from (34) and computing the exact values of **Mean_Load** and **CoV_Load**.

The obtained results are then compared below.

3.2.1 *Log-Normal distribution: exact vs. approximated* **Mean_Load**

The Table 4 and Table 5 below provide both the distribution-free approximation and the exact value of **Mean_Load** under the assumption of log-normality of reserves. The comparison of the two results is summarised in the form of correction factors (i.e. ratio of exact to approximated value) provided in Table 6. By comparing now the correction factors from Table 6 to analogous correction factors for the Lloyd's fine approximation (1) provided in Table 1, we conclude that the distribution-free approximation of **Mean_Load** is confined between the Lloyd's approximation (1) and the exact value, and thus is of generally better quality than the approximation (1).

9 The Brent's method was used to numerically solve the equation.

Table 4: *Approximated* **Mean_Load** *(in %) assuming Log-Normal distribution and* CoV ≤ 50% *(i.e.* $3 < SC \leq 3.25$*).*

CoV_{tr} =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
10%	1.346	1.221	1.095	0.969	0.841	0.712	0.582	0.448	0.311
15%	2.120	1.921	1.723	1.523	1.322	1.119	0.913	0.704	0.488
20%	2.960	2.681	2.402	2.122	1.841	1.557	1.270	0.978	0.678
25%	3.869	3.501	3.134	2.766	2.397	2.026	1.652	1.271	0.881
30%	4.850	4.384	3.919	3.456	2.992	2.526	2.058	1.582	1.096
35%	5.908	5.333	4.762	4.194	3.627	3.059	2.489	1.913	1.324
40%	7.047	6.352	5.664	4.982	4.303	3.626	2.947	2.263	1.565
45%	8.263	7.439	6.625	5.820	5.022	4.227	3.433	2.634	1.821
50%	9.527	8.573	7.632	6.701	5.778	4.861	3.946	3.027	2.094

Table 5: *Exact value of* **Mean_Load** *(in %) assuming Log-Normal distribution and* CoV ≤ 50% *(i.e.* $3 < SC \leq 3.25$*).*

CoV_{tr} =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
10%	1.362	1.236	1.109	0.982	0.853	0.723	0.591	0.456	0.317
15%	2.172	1.970	1.768	1.565	1.360	1.153	0.943	0.728	0.506
20%	3.079	2.793	2.506	2.218	1.927	1.634	1.337	1.033	0.720
25%	4.093	3.711	3.329	2.946	2.560	2.171	1.776	1.373	0.958
30%	5.223	4.735	4.246	3.756	3.263	2.767	2.264	1.752	1.223
35%	6.481	5.872	5.263	4.654	4.043	3.427	2.805	2.171	1.517
40%	7.877	7.133	6.391	5.648	4.904	4.156	3.401	2.632	1.841
45%	9.425	8.529	7.636	6.745	5.853	4.958	4.056	3.139	2.197
50%	11.138	10.071	9.010	7.952	6.897	5.839	4.774	3.695	2.586

Table 6: *Correction factors for* **Mean_Load** *approximation assuming Log-Normal distribution and* CoV ≤ 50% *(i.e.* $3 < SC \leq 3.25$*).*

CoV_{tr} =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
10%	1.01189	1.01235	1.01284	1.01340	1.01403	1.01478	1.01569	1.01688	1.01859
15%	1.02451	1.02544	1.02645	1.02757	1.02884	1.03033	1.03214	1.03448	1.03784
20%	1.04008	1.04161	1.04327	1.04509	1.04714	1.04951	1.05238	1.05607	1.06133
25%	1.05777	1.06003	1.06244	1.06507	1.06801	1.07137	1.07541	1.08056	1.08785
30%	1.07691	1.08001	1.08330	1.08684	1.09074	1.09517	1.10043	1.10708	1.11644
35%	1.09699	1.10107	1.10532	1.10984	1.11476	1.12028	1.12677	1.13490	1.14623
40%	1.11791	1.12301	1.12826	1.13378	1.13970	1.14628	1.15391	1.16337	1.17645
45%	1.14065	1.14655	1.15259	1.15889	1.16561	1.17301	1.18152	1.19200	1.20638
50%	1.16914	1.17469	1.18055	1.18678	1.19353	1.20105	1.20978	1.22057	1.23544

3.2.2 *Log-Normal distribution: exact vs. approximated* **CoV_Load**

Similar analysis of quality of the distribution-free approximation of **CoV_Load** is provided below in Table 7 and Table 8, and summarised in the form of correction factors provided in Table 9. From there we conclude that the quality of **CoV_Load** approximation is as good as that of **Mean_Load** approximation.

Table 7: *Approximated* **CoV_Load** *(in %) assuming Log-Normal distribution and* CoV ≤ 50% *(i.e.* $3 < SC \leq 3.25$*).*

CoV_{tr} =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
10%	13.030	12.080	11.099	10.081	9.018	7.901	6.715	5.436	4.022
15%	14.027	13.009	11.957	10.867	9.729	8.533	7.262	5.890	4.372
20%	15.092	13.996	12.865	11.694	10.473	9.191	7.829	6.359	4.731
25%	16.258	15.070	13.848	12.584	11.269	9.890	8.428	6.852	5.107
30%	17.569	16.268	14.935	13.562	12.137	10.648	9.073	7.379	5.506
35%	19.078	17.636	16.166	14.659	13.104	11.485	9.780	7.953	5.938
40%	20.856	19.231	17.588	15.916	14.201	12.427	10.569	8.587	6.411
45%	22.982	21.123	19.259	17.379	15.467	13.504	11.463	9.300	6.940
50%	25.514	23.366	21.231	19.095	16.943	14.751	12.490	10.114	7.538

Table 8: *Exact value of* **CoV_Load** *(in %) assuming Log-Normal distribution and* CoV ≤ 50% *(i.e.* $3 < SC \leq 3.25$*).*

CoV_{tr} =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
10%	13.246	12.288	11.299	10.270	9.196	8.066	6.864	5.566	4.129
15%	14.496	13.464	12.395	11.284	10.122	8.897	7.593	6.180	4.610
20%	15.896	14.778	13.621	12.417	11.157	9.827	8.408	6.869	5.151
25%	17.461	16.247	14.990	13.682	12.312	10.865	9.319	7.638	5.758
30%	19.209	17.887	16.517	15.092	13.599	12.021	10.333	8.496	6.435
35%	21.159	19.714	18.217	16.660	15.029	13.305	11.460	9.449	7.188
40%	23.331	21.747	20.107	18.402	16.615	14.728	12.707	10.504	8.023
45%	25.751	24.008	22.206	20.333	18.373	16.302	14.086	11.669	8.945
50%	28.444	26.521	24.535	22.472	20.316	18.039	15.606	12.952	9.960

Table 9: *Correction factors for* **CoV_Load** *approximation assuming Log-Normal distribution and* CoV ≤ 50% *(i.e.* $3 < SC \leq 3.25$*).*

CoV_{tr} =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
10%	1.01653	1.01722	1.01797	1.01880	1.01975	1.02087	1.02224	1.02402	1.02657
15%	1.03347	1.03496	1.03658	1.03836	1.04038	1.04273	1.04558	1.04925	1.05450
20%	1.05331	1.05593	1.05874	1.06182	1.06527	1.06925	1.07402	1.08012	1.08878
25%	1.07399	1.07812	1.08252	1.08729	1.09256	1.09857	1.10571	1.11475	1.12744
30%	1.09334	1.09948	1.10594	1.11285	1.12040	1.12890	1.13887	1.15134	1.16865
35%	1.10904	1.11780	1.12689	1.13650	1.14687	1.15838	1.17170	1.18811	1.21061
40%	1.11871	1.13080	1.14323	1.15619	1.16999	1.18510	1.20232	1.22321	1.25140
45%	1.12047	1.13659	1.15302	1.16998	1.18784	1.20713	1.22881	1.25469	1.28900
50%	1.11485	1.13503	1.15561	1.17684	1.19908	1.22292	1.24942	1.28062	1.32129

3.2.3 *Gamma distribution: exact vs. approximated* **Mean_Load**

By comparing correction factors between Log-Normal and Gamma distributions (i.e. comparing Table 12 and Table 6) we conclude that the approximation of **Mean_Load** is of generally better quality for Gamma distribution than for Log-Normal distribution.

Table 10: *Approximated* **Mean_Load** *(in %) assuming Gamma distribution and* CoV ≤ 50% *(i.e. SC = 2).*

CoV_{tr} =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
10%	1.300	1.179	1.058	0.936	0.813	0.689	0.563	0.434	0.301
15%	2.017	1.829	1.641	1.452	1.261	1.068	0.872	0.672	0.466
20%	2.776	2.517	2.257	1.996	1.733	1.468	1.198	0.924	0.641
25%	3.576	3.241	2.906	2.569	2.230	1.888	1.541	1.187	0.824
30%	4.415	4.001	3.585	3.169	2.749	2.327	1.899	1.463	1.014
35%	5.294	4.795	4.295	3.794	3.291	2.784	2.271	1.749	1.213
40%	6.210	5.622	5.034	4.445	3.854	3.259	2.658	2.046	1.418
45%	7.164	6.482	5.802	5.121	4.438	3.751	3.058	2.354	1.631
50%	8.154	7.375	6.597	5.820	5.042	4.260	3.471	2.671	1.851

Table 11: *Exact value of* **Mean_Load** *(in %) assuming Gamma distribution and* CoV ≤ 50% *(i.e. SC = 2).*

CoV_{tr} =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
10%	1.306	1.185	1.063	0.941	0.818	0.693	0.566	0.436	0.303
15%	2.036	1.847	1.658	1.467	1.275	1.080	0.882	0.681	0.473
20%	2.820	2.558	2.296	2.031	1.765	1.496	1.222	0.943	0.655
25%	3.661	3.320	2.979	2.636	2.290	1.940	1.586	1.224	0.851
30%	4.559	4.135	3.709	3.281	2.850	2.415	1.974	1.524	1.060
35%	5.519	5.003	4.487	3.968	3.447	2.920	2.387	1.843	1.282
40%	6.541	5.928	5.315	4.699	4.080	3.457	2.825	2.181	1.518
45%	7.628	6.911	6.194	5.475	4.753	4.025	3.289	2.539	1.767
50%	8.784	7.955	7.127	6.297	5.465	4.627	3.779	2.917	2.031

Table 12: *Correction factors for* **Mean_Load** *approximation assuming Gamma distribution and* CoV ≤ 50% *(i.e. SC = 2).*

CoV_{tr} =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
10%	1.00448	1.00462	1.00478	1.00497	1.00518	1.00543	1.00574	1.00615	1.00673
15%	1.00950	1.00979	1.01012	1.01048	1.01090	1.01140	1.01201	1.01281	1.01396
20%	1.01600	1.01646	1.01697	1.01755	1.01821	1.01898	1.01994	1.02118	1.02297
25%	1.02377	1.02441	1.02512	1.02592	1.02683	1.02790	1.02922	1.03092	1.03337
30%	1.03264	1.03347	1.03438	1.03540	1.03656	1.03793	1.03960	1.04175	1.04484
35%	1.04250	1.04351	1.04462	1.04585	1.04725	1.04889	1.05088	1.05346	1.05714
40%	1.05327	1.05444	1.05572	1.05715	1.05876	1.06064	1.06292	1.06587	1.07006
45%	1.06487	1.06619	1.06762	1.06920	1.07099	1.07307	1.07559	1.07883	1.08344
50%	1.07728	1.07872	1.08026	1.08196	1.08388	1.08610	1.08879	1.09225	1.09715

3.2.4 *Gamma distribution: exact vs. approximated* **CoV_Load**

By comparing correction factors between Log-Normal and Gamma distributions (i.e. comparing Table 15 and Table 9) we conclude that the approximation of **CoV_Load** is of generally better quality for Gamma distribution than for Log-Normal distribution.

Table 13: *Approximated* **CoV_Load** *(in %) assuming Gamma distribution and* CoV ≤ 50% *(i.e. SC = 2).*

CoV_{tr} =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
10%	11.896	11.037	10.148	9.224	8.257	7.238	6.154	4.983	3.686
15%	12.248	11.377	10.474	9.533	8.547	7.506	6.396	5.193	3.856
20%	12.587	11.704	10.788	9.831	8.828	7.766	6.631	5.397	4.022
25%	12.915	12.021	11.092	10.121	9.100	8.018	6.859	5.596	4.184
30%	13.234	12.330	11.388	10.403	9.366	8.265	7.082	5.792	4.343
35%	13.548	12.633	11.679	10.680	9.627	8.507	7.303	5.985	4.501
40%	13.858	12.932	11.967	10.954	9.885	8.747	7.521	6.176	4.658
45%	14.167	13.230	12.253	11.227	10.142	8.986	7.738	6.366	4.814
50%	14.476	13.529	12.539	11.499	10.399	9.225	7.955	6.557	4.971

Table 14: *Exact value of* **CoV_Load** *(in %) assuming Gamma distribution and* CoV ≤ 50% *(i.e. SC = 2).*

CoV_{tr} =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
10%	11.952	11.092	10.200	9.273	8.303	7.281	6.193	5.016	3.714
15%	12.371	11.495	10.587	9.640	8.648	7.599	6.480	5.266	3.916
20%	12.796	11.905	10.980	10.014	8.999	7.924	6.774	5.522	4.124
25%	13.228	12.323	11.380	10.395	9.357	8.256	7.074	5.785	4.338
30%	13.666	12.747	11.788	10.783	9.722	8.595	7.381	6.053	4.557
35%	14.111	13.177	12.201	11.177	10.094	8.940	7.695	6.327	4.781
40%	14.563	13.614	12.622	11.578	10.473	9.292	8.014	6.608	5.011
45%	15.020	14.058	13.049	11.986	10.858	9.650	8.340	6.893	5.246
50%	15.485	14.508	13.482	12.400	11.249	10.014	8.671	7.185	5.485

Table 15: *Correction factors for* **CoV_Load** *approximation assuming Gamma distribution and* CoV ≤ 50% *(i.e. SC = 2).*

CoV_{tr} =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
10%	1.00477	1.00495	1.00514	1.00536	1.00562	1.00592	1.00629	1.00678	1.00750
15%	1.00999	1.01036	1.01076	1.01121	1.01173	1.01234	1.01309	1.01408	1.01550
20%	1.01658	1.01718	1.01784	1.01857	1.01941	1.02040	1.02161	1.02317	1.02542
25%	1.02422	1.02509	1.02604	1.02710	1.02830	1.02971	1.03142	1.03363	1.03678
30%	1.03264	1.03381	1.03509	1.03650	1.03810	1.03996	1.04221	1.04510	1.04919
35%	1.04159	1.04310	1.04474	1.04654	1.04856	1.05089	1.05370	1.05728	1.06231
40%	1.05087	1.05275	1.05477	1.05697	1.05943	1.06226	1.06564	1.06991	1.07588
45%	1.06028	1.06255	1.06498	1.06761	1.07053	1.07387	1.07782	1.08278	1.08965
50%	1.06963	1.07233	1.07519	1.07827	1.08167	1.08552	1.09004	1.09567	1.10340

3.2.5 *Inverse-Gamma distribution: exact vs. approximated* **Mean_Load**

By comparing correction factors between Log-Normal, Gamma and Inverse-Gamma distributions (i.e. comparing Table 12, Table 6 and Table 18) we conclude that the approximation of **Mean_Load** is of marginally lower quality for Inverse-Gamma distribution than for Log-Normal and Gamma distributions.

Table 16: *Approximated* **Mean_Load** *(in %) assuming Inverse-Gamma distribution and* CoV ≤ 50% *(i.e.* 4 < *SC* < 5.3*).*

CoV_{tr} =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
10%	1.393	1.263	1.132	1.001	0.869	0.736	0.600	0.463	0.321
15%	2.228	2.018	1.807	1.596	1.384	1.171	0.955	0.735	0.510
20%	3.167	2.863	2.560	2.258	1.955	1.651	1.345	1.034	0.716
25%	4.228	3.814	3.402	2.994	2.587	2.181	1.774	1.362	0.942
30%	5.444	4.894	4.354	3.821	3.293	2.770	2.247	1.723	1.190
35%	6.783	6.095	5.415	4.744	4.082	3.426	2.775	2.124	1.465

Table 17: *Exact value of* **Mean_Load** *(in %) assuming Inverse-Gamma distribution and* CoV ≤ 50% *(i.e.* $4 < SC < 5.3$*).*

CoV_{tr} =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
10%	1.423	1.291	1.158	1.025	0.891	0.755	0.618	0.477	0.332
15%	2.324	2.108	1.892	1.674	1.455	1.234	1.010	0.781	0.544
20%	3.382	3.067	2.751	2.434	2.116	1.795	1.469	1.137	0.794
25%	4.625	4.191	3.757	3.323	2.887	2.449	2.005	1.553	1.087
30%	6.083	5.508	4.934	4.361	3.787	3.211	2.629	2.037	1.428
35%	7.798	7.052	6.311	5.572	4.835	4.096	3.353	2.598	1.824

Table 18: *Correction factors for* **Mean_Load** *approximation assuming Inverse-Gamma distribution and* CoV ≤ 50% *(i.e.* $4 < SC < 5.3$*).*

CoV_{tr} =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
10%	1.02142	1.02229	1.02324	1.02429	1.02548	1.02688	1.02858	1.03079	1.03396
15%	1.04296	1.04480	1.04679	1.04896	1.05139	1.05421	1.05760	1.06195	1.06816
20%	1.06798	1.07121	1.07462	1.07829	1.08233	1.08693	1.09239	1.09932	1.10910
25%	1.09366	1.09887	1.10424	1.10988	1.11596	1.12273	1.13061	1.14043	1.15410
30%	1.11751	1.12540	1.13334	1.14148	1.15002	1.15931	1.16986	1.18272	1.20030
35%	1.14970	1.15699	1.16533	1.17451	1.18454	1.19562	1.20826	1.22359	1.24432

3.2.6 *Inverse-Gamma distribution: exact vs. approximated* **CoV_Load**

By comparing correction factors between Log-Normal, Gamma and Inverse-Gamma distributions (i.e. comparing Table 15, Table 9 and Table 21) we conclude that the approximation of **CoV_Load** is of marginally lower quality for Inverse-Gamma distribution than for Log-Normal and Gamma distributions.

Table 19: *Approximated* **CoV_Load** *(in %) assuming Inverse-Gamma distribution and* CoV ≤ 50% *(i.e.* $4 < SC < 5.3$*).*

CoV_{tr} =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
10%	14.258	13.205	12.120	10.997	9.829	8.604	7.306	5.911	4.373
15%	16.089	14.887	13.653	12.380	11.060	9.680	8.222	6.657	4.934
20%	18.291	16.887	15.455	13.988	12.476	10.904	9.252	7.488	5.553
25%	21.136	19.430	17.712	15.972	14.197	12.370	10.468	8.455	6.263
30%	25.168	22.949	20.764	18.596	16.427	14.233	11.985	9.638	7.118
35%	31.404	28.263	25.257	22.361	19.545	16.772	14.002	11.176	8.204

Table 20: *Exact value of* **CoV_Load** *(in %) assuming Inverse-Gamma distribution and* CoV ≤ 50% *(i.e.* $4 < SC < 5.3$*).*

CoV_{tr} =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
10%	14.702	13.635	12.533	11.391	10.200	8.948	7.619	6.185	4.598
15%	17.062	15.835	14.570	13.258	11.891	10.454	8.928	7.279	5.448
20%	19.935	18.510	17.042	15.522	13.940	12.278	10.512	8.604	6.481
25%	23.465	21.787	20.062	18.282	16.431	14.491	12.433	10.211	7.737
30%	27.846	25.838	23.783	21.669	19.479	17.191	14.770	12.162	9.262
35%	33.365	30.915	28.420	25.868	23.238	20.504	17.626	14.538	11.115

Table 21: *Correction factors for* **CoV_Load** *approximation assuming Inverse-Gamma distribution and* CoV ≤ 50% *(i.e.* $4 < SC < 5.3$*).*

	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
CoV_{tr} =									
10%	1.03119	1.03260	1.03414	1.03584	1.03777	1.04001	1.04275	1.04629	1.05136
15%	1.06044	1.06368	1.06716	1.07095	1.07518	1.08005	1.08590	1.09338	1.10402
20%	1.08992	1.09612	1.10265	1.10965	1.11732	1.12598	1.13621	1.14909	1.16717
25%	1.11019	1.12129	1.13269	1.14462	1.15739	1.17148	1.18772	1.20773	1.23529
30%	1.10641	1.12592	1.14541	1.16521	1.18578	1.20780	1.23243	1.26187	1.30126
35%	1.06243	1.09382	1.12525	1.15683	1.18899	1.22253	1.25888	1.30082	1.35488

The following two subsections provide the distribution-free approximations of ENID load tabulated for different levels of CoV_{tr}, probability truncation point p and admissible SC ratio.

3.2.7 *Distribution-free approximation of* **Mean_Load**

Table 22: **Mean_Load** *approximation (in %) under* $CoV_{tr} = 10\%$.

	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
SC =									
2.0	1.300	1.179	1.058	0.936	0.813	0.689	0.563	0.434	0.301
2.2	1.309	1.188	1.066	0.943	0.819	0.694	0.566	0.437	0.303
2.4	1.318	1.196	1.073	0.949	0.824	0.698	0.570	0.440	0.305
2.6	1.327	1.204	1.080	0.956	0.830	0.703	0.574	0.442	0.307
2.8	1.336	1.212	1.087	0.962	0.835	0.708	0.578	0.445	0.309
3.0	1.345	1.220	1.095	0.968	0.841	0.712	0.581	0.448	0.311
3.2	1.354	1.228	1.102	0.975	0.846	0.717	0.585	0.451	0.313
3.4	1.363	1.236	1.109	0.981	0.852	0.721	0.589	0.454	0.315
3.6	1.372	1.244	1.116	0.987	0.857	0.726	0.592	0.457	0.317
3.8	1.381	1.252	1.123	0.993	0.862	0.730	0.596	0.459	0.319
4.0	1.390	1.260	1.130	0.999	0.868	0.735	0.600	0.462	0.320
4.2	1.399	1.268	1.137	1.006	0.873	0.739	0.603	0.465	0.322
4.4	1.408	1.276	1.144	1.012	0.878	0.743	0.607	0.467	0.324
4.6	1.417	1.284	1.151	1.018	0.883	0.748	0.610	0.470	0.326
4.8	1.426	1.292	1.158	1.024	0.889	0.752	0.614	0.473	0.328
5.0	1.435	1.300	1.165	1.030	0.894	0.756	0.617	0.475	0.330
5.2	1.444	1.308	1.172	1.036	0.899	0.761	0.621	0.478	0.331

Table 23: **Mean_Load** *approximation (in %) under* $CoV_{tr} = 20\%$.

	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
SC =									
2.0	2.776	2.517	2.257	1.996	1.733	1.468	1.198	0.924	0.641
2.2	2.811	2.548	2.285	2.021	1.754	1.485	1.213	0.934	0.648
2.4	2.846	2.580	2.313	2.045	1.775	1.502	1.226	0.945	0.655
2.6	2.881	2.611	2.340	2.069	1.795	1.519	1.240	0.955	0.663
2.8	2.916	2.642	2.368	2.092	1.816	1.536	1.254	0.966	0.670
3.0	2.951	2.673	2.395	2.116	1.836	1.553	1.267	0.976	0.677
3.2	2.986	2.704	2.422	2.139	1.856	1.570	1.280	0.986	0.683
3.4	3.021	2.735	2.449	2.163	1.875	1.586	1.293	0.996	0.690
3.6	3.056	2.766	2.476	2.186	1.895	1.602	1.306	1.006	0.697
3.8	3.091	2.797	2.503	2.209	1.915	1.618	1.319	1.015	0.704
4.0	3.126	2.827	2.530	2.232	1.934	1.635	1.332	1.025	0.710
4.2	3.161	2.858	2.557	2.255	1.954	1.650	1.345	1.035	0.717
4.4	3.196	2.889	2.583	2.278	1.973	1.666	1.357	1.044	0.723
4.6	3.231	2.920	2.610	2.301	1.992	1.682	1.370	1.053	0.729
4.8	3.266	2.951	2.637	2.324	2.011	1.698	1.382	1.063	0.736
5.0	3.301	2.982	2.664	2.347	2.030	1.713	1.395	1.072	0.742
5.2	3.337	3.012	2.690	2.370	2.049	1.729	1.407	1.081	0.748

Table 24: **Mean_Load** *approximation (in %) under* $\mathrm{CoV}_{\mathrm{tr}}$ = 30%.

	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
SC =									
2.0	4.415	4.001	3.585	3.169	2.749	2.327	1.899	1.463	1.014
2.2	4.492	4.069	3.646	3.221	2.794	2.364	1.928	1.485	1.030
2.4	4.570	4.138	3.706	3.273	2.838	2.400	1.957	1.507	1.045
2.6	4.647	4.206	3.765	3.324	2.881	2.436	1.986	1.529	1.060
2.8	4.724	4.274	3.825	3.375	2.925	2.472	2.015	1.550	1.074
3.0	4.802	4.342	3.884	3.426	2.968	2.507	2.043	1.572	1.089
3.2	4.879	4.410	3.943	3.477	3.010	2.542	2.070	1.592	1.103
3.4	4.957	4.479	4.002	3.527	3.053	2.577	2.098	1.613	1.117
3.6	5.035	4.547	4.061	3.578	3.095	2.611	2.125	1.634	1.131
3.8	5.113	4.615	4.120	3.628	3.137	2.646	2.152	1.654	1.145
4.0	5.191	4.683	4.179	3.678	3.179	2.680	2.179	1.674	1.158
4.2	5.269	4.751	4.237	3.727	3.220	2.714	2.206	1.694	1.172
4.4	5.346	4.818	4.295	3.777	3.261	2.747	2.233	1.714	1.186
4.6	5.422	4.884	4.352	3.825	3.302	2.781	2.259	1.734	1.199
4.8	5.497	4.949	4.409	3.873	3.342	2.814	2.285	1.754	1.213
5.0	5.568	5.012	4.463	3.920	3.382	2.846	2.311	1.773	1.227
5.2	5.636	5.073	4.516	3.966	3.421	2.878	2.337	1.793	1.240

Table 25: **Mean_Load** *approximation (in %) under* $\mathrm{CoV}_{\mathrm{tr}}$ = 40%.

	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
SC =									
2.0	6.210	5.622	5.034	4.445	3.854	3.259	2.658	2.046	1.418
2.2	6.345	5.741	5.138	4.535	3.930	3.322	2.707	2.084	1.444
2.4	6.481	5.861	5.242	4.624	4.005	3.383	2.757	2.121	1.469
2.6	6.616	5.980	5.345	4.712	4.079	3.445	2.805	2.157	1.494
2.8	6.752	6.099	5.449	4.801	4.153	3.505	2.853	2.193	1.519
3.0	6.888	6.218	5.551	4.888	4.227	3.565	2.901	2.229	1.543
3.2	7.024	6.336	5.653	4.975	4.300	3.625	2.948	2.265	1.567
3.4	7.158	6.453	5.754	5.061	4.372	3.684	2.995	2.300	1.591
3.6	7.289	6.567	5.853	5.146	4.443	3.742	3.041	2.335	1.616
3.8	7.415	6.678	5.950	5.228	4.513	3.800	3.087	2.369	1.640
4.0	7.532	6.782	6.041	5.308	4.580	3.856	3.132	2.404	1.664
4.2	7.637	6.877	6.126	5.382	4.644	3.910	3.176	2.438	1.687
4.4	7.724	6.959	6.202	5.450	4.704	3.961	3.218	2.471	1.711
4.6	7.786	7.023	6.264	5.509	4.758	4.009	3.259	2.503	1.735
4.8	7.817	7.063	6.309	5.556	4.804	4.051	3.296	2.535	1.759
5.0	7.808	7.072	6.331	5.587	4.839	4.087	3.330	2.564	1.782

Table 26: **Mean_Load** *approximation (in %) under* $\mathrm{CoV}_{\mathrm{tr}}$ = 50%.

	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
SC =									
2.0	8.154	7.375	6.597	5.820	5.042	4.260	3.471	2.671	1.851
2.2	8.362	7.557	6.756	5.956	5.156	4.354	3.546	2.727	1.889
2.4	8.570	7.740	6.914	6.091	5.270	4.447	3.620	2.783	1.927
2.6	8.778	7.921	7.071	6.225	5.382	4.539	3.693	2.838	1.965
2.8	8.982	8.100	7.226	6.358	5.493	4.630	3.765	2.892	2.002
3.0	9.180	8.274	7.377	6.487	5.602	4.720	3.836	2.946	2.040
3.2	9.366	8.438	7.520	6.610	5.707	4.807	3.906	3.000	2.077
3.4	9.531	8.587	7.652	6.726	5.807	4.890	3.974	3.052	2.114
3.6	9.664	8.711	7.767	6.830	5.898	4.969	4.039	3.104	2.151
3.8	9.750	8.801	7.856	6.915	5.977	5.040	4.100	3.153	2.188
4.0	9.772	8.840	7.907	6.973	6.038	5.099	4.155	3.200	2.224

3.2.8 *Distribution-free approximation of* **CoV_Load**

Table 27: **CoV_Load** *approximation (in %) under* CoV_{tr} = 10%.

SC =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
2.0	11.896	11.037	10.148	9.224	8.257	7.238	6.154	4.983	3.686
2.2	12.115	11.239	10.333	9.390	8.405	7.367	6.264	5.071	3.752
2.4	12.337	11.443	10.519	9.558	8.555	7.498	6.374	5.161	3.818
2.6	12.561	11.649	10.707	9.728	8.705	7.629	6.485	5.250	3.885
2.8	12.787	11.857	10.896	9.898	8.857	7.761	6.596	5.340	3.951
3.0	13.016	12.067	11.087	10.070	9.009	7.893	6.708	5.431	4.019
3.2	13.246	12.279	11.280	10.244	9.163	8.027	6.821	5.522	4.086
3.4	13.480	12.493	11.475	10.419	9.318	8.162	6.935	5.613	4.154
3.6	13.715	12.709	11.671	10.595	9.474	8.297	7.049	5.705	4.221
3.8	13.953	12.927	11.869	10.773	9.631	8.434	7.164	5.797	4.290
4.0	14.194	13.147	12.068	10.952	9.790	8.571	7.279	5.890	4.358
4.2	14.437	13.369	12.270	11.133	9.949	8.709	7.396	5.984	4.427
4.4	14.682	13.594	12.473	11.315	10.110	8.848	7.513	6.077	4.496
4.6	14.930	13.821	12.679	11.499	10.272	8.989	7.630	6.172	4.566
4.8	15.181	14.050	12.886	11.684	10.436	9.130	7.749	6.267	4.635
5.0	15.435	14.281	13.095	11.871	10.601	9.272	7.868	6.362	4.705
5.2	15.691	14.515	13.306	12.060	10.767	9.415	7.988	6.458	4.776

Table 28: **CoV_Load** *approximation (in %) under* CoV_{tr} = 20%.

SC =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
2.0	12.587	11.704	10.788	9.831	8.828	7.766	6.631	5.397	4.022
2.2	13.041	12.121	11.167	10.173	9.131	8.030	6.853	5.577	4.155
2.4	13.505	12.547	11.554	10.521	9.439	8.297	7.079	5.759	4.289
2.6	13.979	12.981	11.948	10.875	9.752	8.568	7.307	5.942	4.425
2.8	14.463	13.424	12.350	11.235	10.070	8.844	7.539	6.128	4.562
3.0	14.959	13.876	12.760	11.601	10.393	9.123	7.774	6.316	4.701
3.2	15.465	14.338	13.177	11.975	10.722	9.407	8.012	6.507	4.841
3.4	15.983	14.810	13.603	12.355	11.056	9.695	8.253	6.699	4.982
3.6	16.514	15.293	14.038	12.742	11.396	9.988	8.497	6.894	5.125
3.8	17.057	15.786	14.481	13.136	11.742	10.285	8.745	7.092	5.269
4.0	17.613	16.290	14.934	13.539	12.094	10.586	8.996	7.291	5.415
4.2	18.183	16.805	15.396	13.948	12.452	10.893	9.251	7.494	5.563
4.4	18.768	17.333	15.868	14.366	12.817	11.205	9.510	7.699	5.712
4.6	19.367	17.873	16.351	14.793	13.188	11.521	9.772	7.907	5.863
4.8	19.981	18.426	16.844	15.227	13.566	11.843	10.039	8.117	6.016
5.0	20.612	18.992	17.348	15.671	13.951	12.171	10.309	8.331	6.171
5.2	21.259	19.572	17.863	16.124	14.343	12.504	10.584	8.547	6.328

Table 29: **CoV_Load** *approximation (in %) under* CoV_{tr} = 30%.

SC =	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
2.0	13.234	12.330	11.388	10.403	9.366	8.265	7.082	5.792	4.343
2.2	13.944	12.980	11.978	10.933	9.834	8.671	7.425	6.067	4.547
2.4	14.678	13.651	12.587	11.478	10.315	9.087	7.775	6.348	4.754
2.6	15.439	14.345	13.214	12.038	10.809	9.513	8.132	6.634	4.964
2.8	16.227	15.062	13.860	12.615	11.316	9.950	8.497	6.925	5.178
3.0	17.045	15.804	14.528	13.209	11.836	10.397	8.870	7.223	5.396
3.2	17.894	16.572	15.217	13.820	12.371	10.855	9.252	7.526	5.617
3.4	18.775	17.368	15.930	14.451	12.921	11.326	9.642	7.836	5.843
3.6	19.691	18.193	16.666	15.101	13.487	11.808	10.042	8.152	6.073
3.8	20.643	19.048	17.427	15.772	14.069	12.303	10.451	8.475	6.308
4.0	21.634	19.935	18.215	16.463	14.668	12.812	10.870	8.805	6.547
4.2	22.664	20.855	19.030	17.177	15.285	13.334	11.300	9.143	6.792
4.4	23.734	21.809	19.873	17.914	15.919	13.870	11.740	9.489	7.041
4.6	24.846	22.798	20.745	18.675	16.573	14.421	12.192	9.842	7.296
4.8	25.998	23.821	21.645	19.458	17.246	14.987	12.655	10.204	7.557
5.0	27.187	24.876	22.573	20.266	17.938	15.569	13.130	10.575	7.823
5.2	28.411	25.962	23.528	21.095	18.648	16.166	13.617	10.955	8.096

Table 30: **CoV_Load** *approximation (in %) under* $\text{CoV}_{\text{tr}} = 40\%$.

	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
SC =									
2.0	13.858	12.932	11.967	10.954	9.885	8.747	7.521	6.176	4.658
2.2	14.851	13.839	12.788	11.690	10.534	9.308	7.992	6.554	4.936
2.4	15.893	14.789	13.645	12.455	11.207	9.888	8.478	6.942	5.222
2.6	16.989	15.784	14.540	13.251	11.905	10.488	8.979	7.342	5.515
2.8	18.142	16.827	15.476	14.081	12.630	11.110	9.496	7.753	5.815
3.0	19.356	17.921	16.454	14.945	13.384	11.753	10.030	8.176	6.124
3.2	20.634	19.069	17.477	15.847	14.167	12.420	10.581	8.612	6.442
3.4	21.979	20.274	18.547	16.787	14.981	13.111	11.152	9.062	6.768
3.6	23.391	21.537	19.666	17.767	15.827	13.828	11.741	9.526	7.104
3.8	24.870	22.855	20.832	18.787	16.706	14.571	12.352	10.005	7.451
4.0	26.406	24.226	22.043	19.845	17.617	15.340	12.982	10.500	7.808
4.2	27.987	25.639	23.293	20.939	18.559	16.134	13.634	11.010	8.177
4.4	29.589	27.079	24.573	22.061	19.528	16.953	14.306	11.537	8.559
4.6	31.178	28.521	25.865	23.201	20.517	17.792	14.996	12.080	8.952
4.8	32.703	29.930	27.145	24.344	21.517	18.647	15.704	12.639	9.359
5.0	34.096	31.256	28.378	25.466	22.514	19.509	16.424	13.212	9.778

Table 31: **CoV_Load** *approximation (in %) under* $\text{CoV}_{\text{tr}} = 50\%$.

	p = 0.95	p = 0.955	p = 0.96	p = 0.965	p = 0.97	p = 0.975	p = 0.98	p = 0.985	p = 0.99
SC =									
2.0	14.476	13.529	12.539	11.499	10.399	9.225	7.955	6.557	4.971
2.2	15.787	14.723	13.617	12.462	11.246	9.955	8.567	7.047	5.332
2.4	17.183	15.990	14.757	13.475	12.135	10.719	9.204	7.556	5.705
2.6	18.671	17.335	15.962	14.543	13.067	11.517	9.869	8.084	6.092
2.8	20.256	18.762	17.236	15.668	14.046	12.352	10.561	8.632	6.492
3.0	21.940	20.274	18.582	16.853	15.074	13.226	11.284	9.203	6.908
3.2	23.718	21.868	19.997	18.096	16.150	14.140	12.037	9.798	7.340
3.4	25.572	23.531	21.475	19.395	17.274	15.093	12.823	10.417	7.790
3.6	27.468	25.240	23.001	20.738	18.439	16.082	13.639	11.061	8.259
3.8	29.348	26.955	24.544	22.108	19.634	17.102	14.484	11.729	8.747
4.0	31.122	28.608	26.060	23.474	20.840	18.142	15.352	12.421	9.255

NOTES ON IMPLEMENTATION It is one of the key aims of the paper to provide quick and practical approximations of ENID load for reserve mean and variability. The implementation of ENID load approximations may well be done in a 'standard formula' style. Below outlines the steps of the proposed implementation.

1. The table of correction factors is pre-computed for each pair of coefficient of variation CoV_{tr} and probability truncation point p separately for each parametric distribution $F \in \mathcal{SSP}'$. They indicate the factor by which the distribution-free approximation needs to be adjusted to arrive at the true exact value of ENID loads **Mean_Load** and **CoV_Load** for a given parametric distribution F.

2. For a given reserve X, for which the shape of its risk profile F_X is characterised by initial proxy information based on observable loss events up to probability truncation point p, i.e. its truncated coefficient of variation $\text{CoV}_{\text{tr}}(X)$ and skewness $\gamma_{tr}(X)$, compute $SC_X = \frac{\gamma_{tr}}{\text{CoV}_{\text{tr}}}$ ratio and compare it against the SC ratio for $\mathcal{SSP}'$ parametric distributions at $\text{CoV}_{\text{tr}}(X)$. This will allow to identify the relative location of the reserve risk profile F_X with respect to the four parametric distributions from $\mathcal{SSP}'$. Let F_X be located between two known parametric distributions F_1 and F_2 from $\mathcal{SSP}'$.

3. For given truncated coefficient of variation $\text{CoV}_{\text{tr}}(X)$ and probability truncation point p identify the corresponding correction factors for the two parametric distributions F_1 and F_2 from $\mathcal{SSP}'$ adjacent to F_X, and then use them to interpolate the correction factors applicable to the reserve risk profile F_X. Here, the interpolation is done in relation to proximity of $SC_X(\text{CoV}_X)$ ratio to analogous ratios of F_1 and F_2 at $\text{CoV}_{\text{tr}}(X)$.

4. Compute the distribution-free approximation of ENID load of reserve risk profile F_X at (CoV_X, p, SC_X) and adjust it by the correction factors obtained in the preceding step using the interpolation.

Practical example. Consider a reserve X with its truncated coefficient of variation $\text{CoV}_{\text{tr}}(X) = 30\%$ and implied (based on observable loss information) SC ratio of 4 assuming the reserve is formed based on the loss events with the return period of up to 20 years, i.e. $p = 0.95$. It follows from here that the given reserve risk profile is confined between Log-Normal and Inverse-Gamma distributions, as for a given level of $\text{CoV}_{\text{tr}}(X)$ at 30%

$$SC_{\text{Log-Normal}} = 3 + 0.3^2 = \mathbf{3.09} < SC_X = \mathbf{4} < SC_{\text{Inv-Gamma}} = \frac{4}{1 - 0.3^2} \sim \mathbf{4.4}.$$

Using the assumptions $\text{CoV}_{\text{tr}}(X) = 30\%$ and $p = 0.95$ we then read

- Table 6 and Table 9 to find the correction factors f_1 and g_1 for **Mean_Load** and **CoV_Load** respectively for the Log-Normal distribution; and
- Table 18 and Table 21 to find the correction factors f_2 and g_2 for **Mean_Load** and **CoV_Load** respectively for the Inverse-Gamma distribution.

Those correction factors are $f_1 = 1.07691$, $g_1 = 1.09334$, $f_2 = 1.11751$ and $g_2 = 1.10641$. Using the linear interpolation we estimate the correction factors f and g to be applied to the distribution-free approximations of ENID load for the reserve X mean and variability:

$$\begin{aligned} f &= 1.10511, \\ g &= 1.10242. \end{aligned}$$

The initial values of the distribution-free approximations of ENID load for the reserve mean and variability are read from Table 24 and Table 29 respectively for $\text{CoV}_{\text{tr}}(X) = 30\%$ and $p = 0.95$:

$$\begin{aligned} \textbf{Mean\_Load} &= 5.191\%, \\ \textbf{CoV\_Load} &= 21.634\%. \end{aligned}$$

Their adjusted values are then equal to

$$\begin{aligned} \textbf{Mean\_Load}' &= f \cdot \textbf{Mean\_Load} = 5.737\%, \\ \textbf{CoV\_Load}' &= g \cdot \textbf{CoV\_Load} = 23.850\%. \end{aligned}$$

4 CONCLUSIONS

Under the upcoming Solvency II insurers will be required to adjust their technical provisions for binary events or events not in data (ENID). This research originated from a simple analysis of so called Lloyd's approximations of Binary Events loading for reserve mean under the assumption of log-normality of reserve values. In this paper, we analytically derived the Lloyd's approximations from first principles, analysed their quality and identified the finest approximation. To make a step further in this research we relaxed the log-normality assumptions of reserves and developed a distribution-free approach that would allow one to practically estimate the ENID load for reserve mean and variability assuming general (non-normal) characteristics of the reserve risk profile. Here, the reserve risk profile is assumed to be fully characterised by its variability and skewness per unit of variability measure (i.e. SC ratio). The distribution-free ENID load approximations are derived and tabulated for a wide spectrum of practically feasible reserve risk profiles with the SC ratio ranging from 2 (equivalent to moderately skewed distributions like Gamma and Inverse-Gaussian) to 5.5 (equivalent to extremely skewed distributions like Inverse-Gamma).

ACKNOWLEDGEMENT

The author[10] wishes to thank Nikolaos Petalas (Inter Hannover, London UK), Anam Khan (PwC, London UK) and Vivek Syal (PwC, London UK) for their review and invaluable comments.

REFERENCES

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, New York, 9 edition, 1972.

[2] N. G. de Bruijn. *Asymtotic Methods in Analysis*. North-Holland Publishing Co., Amsterdam, 1958.

[3] R. A. Fisher and E. A. Cornish. The percentile points of distributions having known cumulants. *Technometrics*, 5:63–69, 1960.

[4] A. I. Fleishman. A method for simulating non-normal distributions. *Psychometrika*, 43:521–532, 1978.

[5] GIRO-2013. Solvency II Technical Provisions for General Insurers. The Institute of Actuaries (UK), 2013. Web link to PDF file.

[6] Christian Kleiber and Samuel Kotz. *Statistical Size Distributions in Economics and Actuarial Sciences*, volume 381 of *Statistical Size Distributions in Economics and Actuarial Sciences*. Wiley-Interscience, New York, USA, 1st edition, 2003.

[7] Samuel Kotz, Norman Johnson, and N Balakrishnan. *Continuous Univariate Distributions*, volume 1. Wiley, New York, USA, 2nd edition, 1994.

[8] LLOYD'S. *Technical Provisions under Solvency II: Detailed Guidance*, 2011. Web link to PDF file.

[9] Albert Marshall and Ingram Olkin. *Life Distributions: Structure of Nonparametric, Semiparametric, and Parametric Families*. Springer Series in Statistics. Springer, New York, USA, 2007.

[10] S. Nadarajah. The exponentiated exponential distribution: a survey. *AStA Advances in Statistical Analysis*, 95:219–251, 2011.

[11] S. Nadarajah and S. Kotz. The exponentiated type distributions. *Acta Applicandae Mathematicae*, 92:97–111, 2006.

[12] Arne Sandström. *Handbook of Solvency for Actuaries and Risk Managers*. Chapman & Hall, UK, 2011.

[13] S. Teukolsky et al. *Numerical Recipes: The Art of Scientific Computing*. Cambridge University Press, 3rd edition, 2007.

[14] Wolfram Documentation Center. Parametric statistical distributions, 2014. Web link.

10 **Disclaimer:** The views and opinions expressed in this article are those of the author and do not reflect the official policy or position of PwC.

Appendix A: Derivation of the k-th truncated moment of Log-Normal distribution

Consider a random variable X that is log-normally distributed with unknown parameters μ and σ, i.e. $X \sim \mathcal{LN}(\mu, \sigma^2)$ with its cumulative distribution function $F_X(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$ and probability density function $f_X(x) = \frac{1}{\sigma x}\varphi\left(\frac{\ln x - \mu}{\sigma}\right)$, where $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ is the probability density of standard normal distribution.

The truncated k-th moment of X, truncated by the condition $\{X \leq b\}$, then equals:

$$
\begin{aligned}
m_{tr}^{(k)} &= \mathbb{E}[X^k | X \leq b] \\
&\overset{(??)}{=} \frac{1}{F_X(b)} \int_0^b x^k f_X(x)\, dx \\
&= \frac{1}{F_X(b)} \int_0^b x^k \frac{1}{\sigma x} \varphi\left(\frac{\ln x - \mu}{\sigma}\right) dx
\end{aligned}
$$

$$
\left| \text{1st change of variables:} \quad u := \frac{\ln x - \mu}{\sigma}\Big|_{-\infty}^{\frac{\ln b - \mu}{\sigma}} \Rightarrow du = \frac{dx}{\sigma x}, \quad x = e^{\mu + \sigma u} \Rightarrow x^k = e^{k\mu + k\sigma u} \right|
$$

$$
\begin{aligned}
&= \frac{1}{F_X(b)} \int_{-\infty}^{\frac{\ln b - \mu}{\sigma}} e^{k\mu + k\sigma u} \varphi(u)\, du, \\
&= \frac{1}{F_X(b)} \int_{-\infty}^{\frac{\ln b - \mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(u^2 - 2k\sigma u - 2k\mu\right)} du, \\
&= \frac{e^{k\mu + \frac{1}{2}k^2\sigma^2}}{F_X(b)} \int_{-\infty}^{\frac{\ln b - \mu}{\sigma}} \varphi(u - k\sigma)\, du,
\end{aligned}
$$

$$
\left| \text{2nd change of variables:} \quad t := u - k\sigma\Big|_{-\infty}^{\frac{\ln b - \mu}{\sigma} - k\sigma} \Rightarrow dt = du \right|
$$

$$
\begin{aligned}
&= \frac{e^{k\mu + \frac{1}{2}k^2\sigma^2}}{F_X(b)} \int_{-\infty}^{\frac{\ln b - \mu}{\sigma} - k\sigma} \varphi(t)\, dt, \\
&= e^{k\mu + \frac{1}{2}k^2\sigma^2} \frac{\Phi\left(\frac{\ln b - \mu}{\sigma} - k\sigma\right)}{\Phi\left(\frac{\ln b - \mu}{\sigma}\right)}. \qquad (35)
\end{aligned}
$$

Appendix B: The derivation of roots of cubic equation in (22)

Consider the following cubic equation with real parameters a, b, c and d:

$$az_\alpha^3 + bz_\alpha^2 + cz_\alpha + d = 0, \tag{36}$$

The roots of the cubic equation (36) can be found using the Cardano's formula (see, e.g., Abramowitz and Stegun [1])

$$\begin{cases} x_1 &= M + N - \frac{b}{3a}, \\ x_2 &= -\frac{M+N}{2} - \frac{b}{3a} + \frac{i\sqrt{3}}{2}(M-N), \\ x_3 &= -\frac{M+N}{2} - \frac{b}{3a} - \frac{i\sqrt{3}}{2}(M-N), \end{cases} \tag{37}$$

where

$$M = \sqrt[3]{P + \sqrt{P^2 + Q^3}}, \tag{38}$$

$$N = \sqrt[3]{P - \sqrt{P^2 + Q^3}}, \tag{39}$$

where

$$P = \frac{9abc - 27a^2d - 2b^3}{54a^3}, \tag{40}$$

$$Q = \frac{3ac - b^2}{9a^2}. \tag{41}$$

The existence of real roots of the cubic equation (36) and their quantity are dependent on the sign of cubic discriminant

$$D = P^2 + Q^3. \tag{42}$$

We consider three cases of D:

1. if $D > 0$, then there exists only one real root and it is equal to $M + N - \frac{b}{3a}$;
2. if $D = 0$, then all three roots are real, and at least two are the same and equal to $-\frac{M+N}{2} - \frac{b}{3a}$; and
3. if $D < 0$, then all three roots are real and unequal.

In the latter case, when $D < 0$, the three real roots can also be expressed trigonometrically:

$$\begin{cases} x_1 &= 2\sqrt{-Q}\cos\left(\frac{\varphi}{3}\right) - \frac{b}{3a}, \\ x_2 &= 2\sqrt{-Q}\cos\left(\frac{\varphi}{3} + \frac{2\pi}{3}\right) - \frac{b}{3a}, \\ x_3 &= 2\sqrt{-Q}\cos\left(\frac{\varphi}{3} + \frac{4\pi}{3}\right) - \frac{b}{3a}, \end{cases} \tag{43}$$

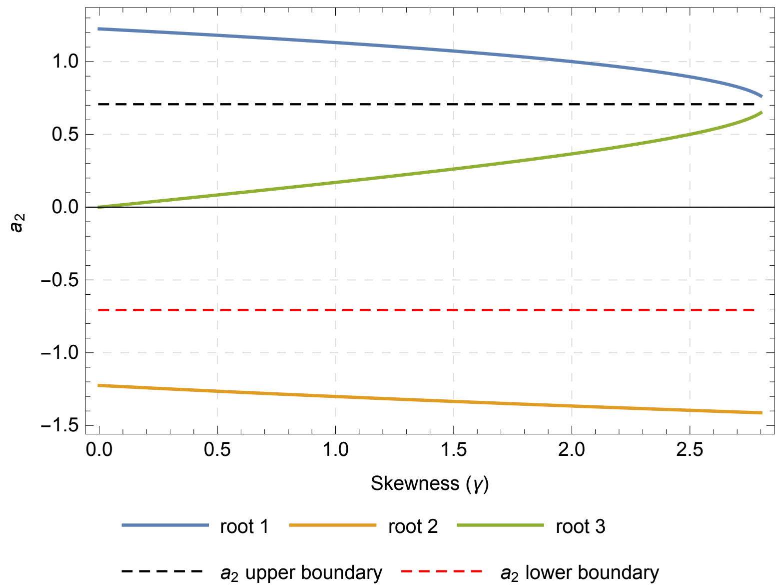
where $\varphi = \arccos\left(\frac{P}{\sqrt{-Q^3}}\right)$.

Now, let us consider the specific cubic equation defined in (22). In this case we have

$$\begin{aligned} a &= 4; \\ b &= 0; \\ c &= -6; \\ d &= \gamma; \\ P &= -\frac{d}{2a} = -\frac{\gamma}{8}; \\ Q &= \frac{c}{3a} = -\frac{1}{2}; \\ D &= \frac{\gamma^2}{64} - \frac{1}{8}; \text{ and} \\ \varphi &= \arccos\left(-\frac{\gamma}{2\sqrt{2}}\right). \end{aligned}$$

Its discriminant D is negative for $\gamma < 2\sqrt{2}$, indicating that there are three real roots as defined in (43): 'root 1' x_1, 'root 2' x_2 and 'root 3' x_3. By analysing those roots as functions of γ we could eliminate the roots which fall outside the interval of admissible values of a_2, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Figure 5: *Real roots of cubic equation in (22).*



The Figure 5 above shows that there is only one suitable root for a_2 and that is 'root 3' which equals

$$\sqrt{2}\cos\left(\frac{\varphi}{3} + \frac{4\pi}{3}\right).$$

Appendix C: Truncated moments of standard normal variable

Consider the n-th truncated moment of Z, I_n

$$\begin{aligned} I_n &= \mathbb{E}\left[Z^n \,|\, c \le Z \le d\right], \; n \ge 0. \\ &= \frac{1}{\Phi(d) - \Phi(c)} \int_c^d x^n \varphi(x)\, dx. \end{aligned} \tag{44}$$

By using the property of the density of standard normal distribution

$$\varphi'(x) = -x\, \varphi(x),$$

and taking the integral in (44) by parts we obtain the following recursive formula

$$\begin{aligned} I_n &= \frac{1}{\Phi(d) - \Phi(c)} \int_c^d x^n \varphi(x)\, dx \\ &= -\frac{1}{\Phi(d) - \Phi(c)} \int_c^d x^{n-1}\, d\varphi(x) \\ &= -\frac{1}{\Phi(d) - \Phi(c)} \left(x^{n-1}\varphi(x)\big|_c^d - (n-1) \int_c^d x^{n-2}\varphi(x)\, dx \right) \\ &= -\frac{d^{n-1}\varphi(d) - c^{n-1}\varphi(c)}{\Phi(d) - \Phi(c)} + (n-1) I_{n-2}, \; n \ge 2. \end{aligned} \tag{45}$$

The initial two terms I_0 and I_1 of the recursive formula (45) can be trivially computed as follows

$$\begin{cases} I_0 = 1, \text{ and} \\ I_1 = -\frac{\varphi(d) - \varphi(c)}{\Phi(d) - \Phi(c)}. \end{cases} \tag{46}$$

ABOUT AUTHORS

YURIY KRVAVYCH is a Senior Manager of Actuarial and Insurance Management Solutions at PwC in London. Prior to joining PwC, worked for six years in Internal Capital Modelling at Insurance Australia Group (IAG) in Sydney. Prior to this, worked for seven years as an actuary in general insurance industry and gained extensive experience in pricing, reserving, forecasting and risk management modelling. He holds PhD in Mathematics from Kiev University and PhD in Actuarial Science from the University of New South Wales (UNSW). Author of several scientific publications, Yuriy is a frequent speaker at actuarial and mathematical conferences.

This article was produced by pdfLaTeX on MikTeX platform, using classicarticle typographic style.