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General measure for optimal portfolio risk management

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Abstract

In this paper we offer a novel generalized optimal portfolio selection measure which is a ratio of a function of a linear functional and a function of a quadratic functional. This new measure incorporates as special cases many important measures such as the mean variance, Sharpe, tail conditional expectation, tail mean variance, to mention a few. We provide an explicit solution to optimal portfolio selection based on this new measure and thus offer a solution to all the members of the family defined by the new measure discussed here.

Keywords: Function of quadratic functional; Fractional programming; Linear constraints; Mean-Variance model; Optimal portfolio selection; Optimization; Portfolio selection; Sharpe ratio; Tail conditional expectation; Tail mean variance

1 Introduction

The pioneering work of Markowitz (1952) (see Steinbach (2001) [6], and Elton and Gruber (1987) [1]) on optimal portfolio selection (OPS), introduced the classical mean variance (MV) OPS, which has been studied extensively. The MV measure is founded on the maximization of the mean variance risk measure

$$MV(R) := E(R) - \lambda Var(R), \tag{1}$$

where R is the portfolio return, $R = \mathbf{x}^T\mathbf{X}$. $\mathbf{x}$ and $\mathbf{X}$ are a $n \times 1$ vectors of weights and returns, respectively and $\mathbf{1}^T\mathbf{x} = 1$, where $\mathbf{1}$ is vector of n ones. $E(R) = \mathbf{x}^T\boldsymbol{\mu}$, where $\boldsymbol{\mu}^T = (\mu_1, \mu_2, ..., \mu_n)$, is the vector of expected returns and $Var(R) = \mathbf{x}^T\Sigma\mathbf{x}$, where Σ is the $n \times n$ covariance matrix of $\mathbf{X}$.

Another key approach in the OPS theory is the maximization of the Sharpe ratio (see, for instance, Sharpe (1998) [5]), which was introduced in Sharpe (1966) [4], and has the form

$$S := \frac{E(R) - R_f}{\sqrt{Var(R)}}, \tag{2}$$

where R_f is the risk free rate.

Both the MV and the Sharpe measures incorporate, differently, the mean and variance of the return R. In this paper we offer a novel generalized measure, which allows flexible functional platforms for incorporating $E(R)$ and $Var(R)$:

$$\mathcal{F}(\mathbf{x}) = \mathcal{F}(\mathbf{x}; t, p, s) = t\left(\frac{p(E(R))}{s(Var(R))}\right). \tag{3}$$

$t(x)$, $p(x)$ and $s(x)$ are differentiable functions, where $t(x)$ is strictly increasing, i.e., $t'(x) \neq 0$, $p(x)$ is a concave function and $s(x)$, defined on $[0, \infty)$, is positive on $(0, \infty)$ and meets the following condition (denoted by (C)):

(C) The function $v(x) = s(ax^2 + 2qx + r)$ is strongly convex on R for any numbers $a > 0$, q, and r such that $q^2 - ar < 0$.

The new generalized measure suggested here, includes, as special cases, key measures employed in risk management:

Special case 1: Mean Variance (MV) model. In this case $t(x) = \ln x$, $p(x) = e^{-x}$ and $s(x) = e^{-\lambda x}$, $\lambda > 0$, and the goal function (3) has the form

$$\mathcal{F}\left(\mathbf{x}; \ln x, e^{-x}, e^{-\lambda x}\right) = -\mathbf{x}^T\boldsymbol{\mu} + \boldsymbol{\lambda}\mathbf{x}^T\Sigma\mathbf{x}, \tag{4}$$

Special case 2: Sharpe ratio (SR). For $t(x) = x$, $p(x) = x - R_f$ and $s(x) = \sqrt{x}$, (3) takes the form of the Sharpe ratio (2)

$$\mathcal{F}\left(\mathbf{x}; x, x - R_f, \sqrt{x}\right) = \frac{\mathbf{x}^T\boldsymbol{\mu} - R_f}{\sqrt{\mathbf{x}^T\Sigma\mathbf{x}}}. \tag{5}$$

Special case 3: Tail conditional expectation (TCE) (see Landsman (2008) [2]). In the case $t(x) = \ln x$, $p(x) = e^{-x}$ and $s(x) = e^{-\lambda\sqrt{x}}$, $\lambda > 0$, (3) has the form

$$\begin{aligned} \mathcal{F}\left(\mathbf{x}; \ln x, e^{-x}, e^{-\lambda_q\sqrt{x}}\right) &= -\mathbf{x}^T\boldsymbol{\mu}+\lambda_q\sqrt{\mathbf{x}^T\Sigma\mathbf{x}}, \\ \lambda_q &> 0, \end{aligned} \tag{6}$$

where λ_q is a constant under quantile $q \in (0, 1)$.

Special case 4: Tail mean variance (TMV) (see Landsman (2010) [3]). In the case $t(x) = \ln x$, $p(x) = e^{-x}$ and $s(x) = e^{-\lambda_1\sqrt{x}-\lambda_2 x}$, $\lambda_1, \lambda_2 > 0$, (3) has the form

$$\mathcal{F}\left(\mathbf{x}; \ln x, e^{-x}, e^{-\lambda_1\sqrt{x}-\lambda_2 x}\right) = -\mathbf{x}^T\boldsymbol{\mu}+\lambda_1\sqrt{\mathbf{x}^T\Sigma\mathbf{x}} + \lambda_2\mathbf{x}^T\Sigma\mathbf{x}. \tag{7}$$

Special case 5: Coefficient of variation measure. For $t(x) = x$, $p(x) = x^{-1}$ and $s(x) = x^{-1/2}$, (3) takes the form

$$\mathcal{F}\left(\mathbf{x}; x, x^{-1}, x^{-1/2}\right) = \frac{\sqrt{\mathbf{x}^T\Sigma\mathbf{x}}}{\mathbf{x}^T\boldsymbol{\mu}}. \tag{8}$$

The generalized measure we introduce is based on the maximization of (3) subject to the linear constraints

$$B\mathbf{x} = \mathbf{c},\ \mathbf{c} \neq \mathbf{0}. \tag{9}$$

The motivation of such a system of linear constraints arises from models such as the MV one, where the linear constraint $\mathbf{1}^T\mathbf{x} = 1$ consider the sum of the weights of the portfolio to be equal to one with positive and negative values that may describe long and short positions, respectively.

We show that $\mathbf{x}^*$, the solution of the maximization problem, is given by

$$\mathbf{x}^* = \Sigma^{-1}B^T(B\Sigma^{-1}B^T)^{-1}\mathbf{c}+w^*\left(\Delta^T Q^{-1}, -\Delta^T Q^{-1}D_{12}\right)^T,$$

where w^* is obtained explicitly in terms of functions $t(x), p(x)$ and $s(x)$ and matrices Σ, B, D, Q and vector Δ are given in [3]. We also outline the expression of $\mathbf{x}^*$ for special cases.

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