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General approach to the optimal portfolio risk management

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Outline

1. Mean-variance risk measure

2. Generalization based on Sharpe ratio

3. Main results on optimal portfolio selection (OPS)

4. Important examples: generalized Sharpe ratio and classical Sharpe ratio and OPS

5. Combination of linear and function of square root of quadratic functional

6. Translation-invariant and positive-homogeneous risk measures and OPS

7. Tail mean-variance principle and OPS

-The success of classical MV optimal portfolio theory is mostly because the model reduces to QP, which is quite convenient for the technical realization and for the case when short selling is possible, even provides the analytic solution (Merton (1972))

- We suggest the generalization of MV-model which reduces to more complicated CFP – concave fractional programming

Return on the portfolio $-R$

$$R = x_1 R_1 + x_2 R_2 + \cdots + x_n R_n$$

Mean-Variance approach

$$E(R) - \lambda Var(R) \to \max$$

$$\sum_{i=1}^{n} x_i = 1$$

$$MV(R) = E(-R) + \lambda Var(R) \to \inf$$

Another approach to OPS referenced to Sharpe

$$S = \frac{E(R) - R_f}{\sqrt{Var(R)}} \to \max$$

$R_f - risk\ \ free\ rate$

How to combine these two approaches and generalize maximally ?

We suggest the following functional

$$f = t\left(\frac{p(E(R)}{v(Var(R))}\right) \to \max$$

$$t(x) - strictly\ increased,\ p(x) - concave$$

$$t'(x),\ p'(x) \neq 0$$

Important examples

Generalized Sharpe ratio

$$t(x) = x,\ p(x) = x - R_f,\ v(x) = x^{\beta},\ \beta \geq 1/2$$

$$f = \frac{E(R) - R_f}{(Var(R))^{\beta}} \to \max$$

$$\beta = 1/2 \Rightarrow classical\ Sharpe\ ratio$$

Another important functional

$$t(x) = \log x,\ p(x) = \exp(x),\ v(x) = \exp(\lambda s(x)),\ \lambda > 0$$

$$f = E(R) - \lambda s(Var(R)) \to \max$$

$s(x) -$ defined on $[0, \infty)$ function,

differentiable and positive on $(0, \infty)$

Landsman and Makov (2013)

Important examples

$$f = E(R) - \lambda (Var(R))^{\beta}, \beta \geq 1/2$$

$$\beta = 1 \Rightarrow classical\ Mean - Variance\ principle$$

Standard deviation premium principle

$$f = E(R) - \lambda \sqrt{Var(R)},$$

Tail mean-variance premium principle

$$f = E(-R) + \alpha_1 \sqrt{Var(R)} + \alpha_2 Var(R),$$

Portfolio management

$$\boldsymbol{R} = (R_1, \ldots, R_n)^T$$

$$R = \sum_{j=1}^{n} x_j R_j, \qquad \sum_{j=1}^{n} x_j = 1$$

Main result

$$f = t\left(\frac{p(\boldsymbol{\mu}^T \mathbf{x})}{v(\mathbf{x}^T \Sigma \mathbf{x})}\right) \to \max$$

$$B\boldsymbol{x} = \boldsymbol{c}$$

$$rank(B) = m$$

Partitions

$$\boldsymbol{\mu}^T = (\boldsymbol{\mu}_1^T, \boldsymbol{\mu}_2^T), \boldsymbol{\mu}_1^T = (\mu_1, \ldots, \mu_{n-m}), \boldsymbol{\mu}_2^T = (\mu_{n-m+1}, \ldots, \mu_n)$$

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \qquad B = \begin{pmatrix} B_{21} & B_{22} \end{pmatrix}$$

$$Let\ u_1(x) = \frac{v'(x)}{v(x)}\ and\ u_2(x) = \frac{p'(x)}{p(x)}$$

Theorem . If equation

$$u_1(f_0 + b^2 w^2) w = \frac{1}{2} u_2(\boldsymbol{\mu}_2^T B_{22}^{-1}\boldsymbol{c} + \Delta^T Q^{-1}(D_{12}\Sigma_{22} - \Sigma_{12}) B_{22}^{-1}\boldsymbol{c} + wb^2)$$

has the solution w*. Then the optimization problem has the solution

$$\boldsymbol{x}^* = \mathbf{z}_1 + w^* \mathbf{z}_2$$

where

$$\boldsymbol{z}_1 = \Sigma^{-1} B^T (B \Sigma^{-1} B^T)^{-1} \boldsymbol{c}$$

$$\boldsymbol{z}_2 = (\Delta^T Q^{-1}, -\Delta^T Q^{-1} D_{12})^T$$

Generalized Sharpe ratio

$$f = \frac{E(R) - R_f}{(Var(R))^{\beta}} \to \max$$

$$t(x) = x,\ p(x) = x - R_f,\ v(x) = x^{\beta},\ \beta \geq 1/2$$

$$u_1(x) = \frac{\beta}{x}, u_2(x) = \frac{1}{x - R_f}$$

The main equation reduces to quadratic equation

$$\beta(\boldsymbol{\mu}_2^T B_{22}^{-1}\boldsymbol{c} + \Delta^T Q^{-1}(D_{12}\Sigma_{22} - \Sigma_{12})B_{22}^{-1}\boldsymbol{c} + wb^2 - R_f)w = \frac{1}{2}(f_0 + b^2 w^2)$$

2 solutions: w_1^*, w_2^*

$$w^* = \arg\max(f_{w_1^*}, f_{w_2^*})$$

$$\boldsymbol{x}^* = \mathbf{z}_1 + w^* \mathbf{z}_2$$

Classical Sharpe ratio $\beta = 1/2$

The main equation reduces to equation of 1 degree

$$\frac{1}{2}(\boldsymbol{\mu}_2^T B_{22}^{-1}\boldsymbol{c} + \Delta^T Q^{-1}(D_{12}\Sigma_{22} - \Sigma_{12})B_{22}^{-1}\boldsymbol{c} + wb^2 - R_f)w = \frac{1}{2}(f_0 + b^2 w^2)$$

$$w^* = \frac{f_0}{\boldsymbol{\mu}_2^T B_{22}^{-1}\boldsymbol{c} + \Delta^T Q^{-1}(D_{12}\Sigma_{22} - \Sigma_{12})B_{22}^{-1}\boldsymbol{c} - R_f}$$

$$\boldsymbol{x}^* = \mathbf{z}_1 + w^* \mathbf{z}_2$$

Combination of linear and function of square root of quadratic functional

Recall

$$f = t\left(\frac{p(E(R))}{v(Var(R))}\right) = t\left(\frac{p(\boldsymbol{\mu}^T \mathbf{x})}{v(\mathbf{x}^T \Sigma \mathbf{x})}\right) \to \max$$

If

$$t(x) = \log x,\ p(x) = \exp(x), v(x) = \exp(\lambda s(x)), \lambda > 0$$

$$f = \boldsymbol{\mu}^T \mathbf{x} - \lambda s(\mathbf{x}^T \boldsymbol{\Sigma} \mathbf{x}) \to \max$$

Let $s_1(x) = s'(x) > 0, x \in (0, \infty)$

Theorem . If equation

$$w s_1(f_0 + b^2 w^2) = \frac{1}{2\lambda}$$

has the positive solution $\boldsymbol{w}^*$

this solution is unique and the problem

$$\boldsymbol{f}(\mathbf{x}) = -\boldsymbol{\mu}^T \mathbf{x} + \lambda s(\mathbf{x}^T \boldsymbol{\Sigma} \mathbf{x}) \to \inf$$

$$\boldsymbol{B}\boldsymbol{x} = \boldsymbol{c}$$

has the unique solution

$$\boldsymbol{x}^* = \boldsymbol{z}_1 + w^* \boldsymbol{z}_2$$

Application to the optimal portfolio selection

$$s(x) = \sqrt{x}$$

$$f(\mathbf{x}) = -\boldsymbol{\mu}^T \mathbf{x} + \lambda \sqrt{\mathbf{x}^T \Sigma \mathbf{x}} \to \inf$$

Translation invariant and positive homogeneous risk measures

1) $\rho(X + \alpha) = \rho(X) + \alpha, \quad \alpha - const$

2) $\rho(cX) = c\rho(X), \quad c > 0 - const$

Examples

BASEL II

VaR

$$VaR_q(X) = \inf\{x \mid F_X(x) \geq q\}$$

Tail conditional expectation

$$TCE_q(X) = E(X \mid X > VaR_q(X))$$

Tail VaR, Expected Short Fall (ES), Conditional VaR (CVaR)

STD-premium

$$\rho(X) = E(X) + \lambda STD(X)$$

Distorted risk measures

$$\rho(X) = \int_0^\infty g(\bar{F}(x))dx,$$

Denneberg (1994) and Wang (1996)

Coherent risk measures

Artzner, Delbean, Eber, and Heath (1999)

Application of Theorem 1

$$s(x) = \sqrt{x}$$

$$s_1(x) = s'(x) = \frac{1}{2\sqrt{x}}$$

Recall general equation

$$ws_1(f_0 + b^2w^2) = \frac{1}{2\lambda}$$

Reduces to

$$w \frac{1}{\sqrt{f_0 + b^2 w}} = \frac{1}{\lambda}$$

$$(\lambda^2 - b^2) w^2 = f_0$$

The positive solution

$$w^* = \sqrt{\frac{f_0}{\lambda^2 - b^2}} = \sqrt{\frac{\mathbf{c}^T (B \Sigma^{-1} B^T)^{-1} \mathbf{c}}{\lambda^2 - \mathbf{\Delta}^T Q^{-1} \mathbf{\Delta}}}$$

Exists iff

$$\lambda^2 > b^2 = \mathbf{\Delta}^T Q^{-1} \mathbf{\Delta}$$

Recall the general solution

$$\boldsymbol{x}^* = \boldsymbol{z}_1 + w^* \boldsymbol{z}_2$$

$$w^* = \sqrt{\frac{\boldsymbol{c}^T (B\Sigma^{-1}B^T)^{-1}\boldsymbol{c}}{\lambda^2 - \mathbf{\Delta}^T Q^{-1}\mathbf{\Delta}}}$$

This well conforms with

L. (2008), L. and Makov (2011)

Power function s

$$s(x) = x^{\beta}, \beta \geq \frac{1}{2}$$

$$f(\mathbf{x}) = -\boldsymbol{\mu}^T \mathbf{x} + \lambda (\mathbf{x}^T \boldsymbol{\Sigma} \mathbf{x})^{\beta} \to \inf$$

Condition ©

$$v(x) = s(px^2 + qx + r)$$

$$= (px^2 + qx + r)^{\beta} \rightarrow convex$$

$$q^2 - pr < 0$$

$$v''(x) = (px^2 + 2qx + r)^{\beta-2}$$

$$\times((2\beta - 1)(px^2 + q)^2 - (q^2 - pr)) > 0$$

Application of Theorem 1

$$s(x) = x^{\beta}$$

$$s_1(x) = s'(x) = \beta x^{\beta - 1}$$

General equation

$$ws_1(f_0 + b^2 w^2) = \frac{1}{2\lambda}$$

Reduces to power equation

$$cw^{\frac{1}{1-\beta}} - b^2 w^2 - f_0 = 0 \quad (*)$$

$$where\ c = (2\beta\lambda)^{\frac{1}{1-\beta}}$$

For $\beta = \frac{1}{2}$ (*) reduces

$$w^2(\lambda^2 - b^2) = f_0$$

Then the positive solution

$$w^* = \sqrt{\frac{f_0}{\lambda^2 - b^2}}, \quad \lambda^2 > b^2 = \mathbf{\Delta}^T Q^{-1} \mathbf{\Delta}$$

well conforms with TIPH case.

Another special case $\beta = \frac{3}{4}$

The power equation

$$cw^{\frac{1}{1-\beta}} - b^2 w^2 - f_0 = 0 \quad (*)$$

$$where\ c = (2\beta\lambda)^{\frac{1}{1-\beta}}$$

Reduces to biquadratic equation

$$cw^4 - b^2 w^2 - f_0 = 0$$

where $c = (1.5\lambda)^4$.

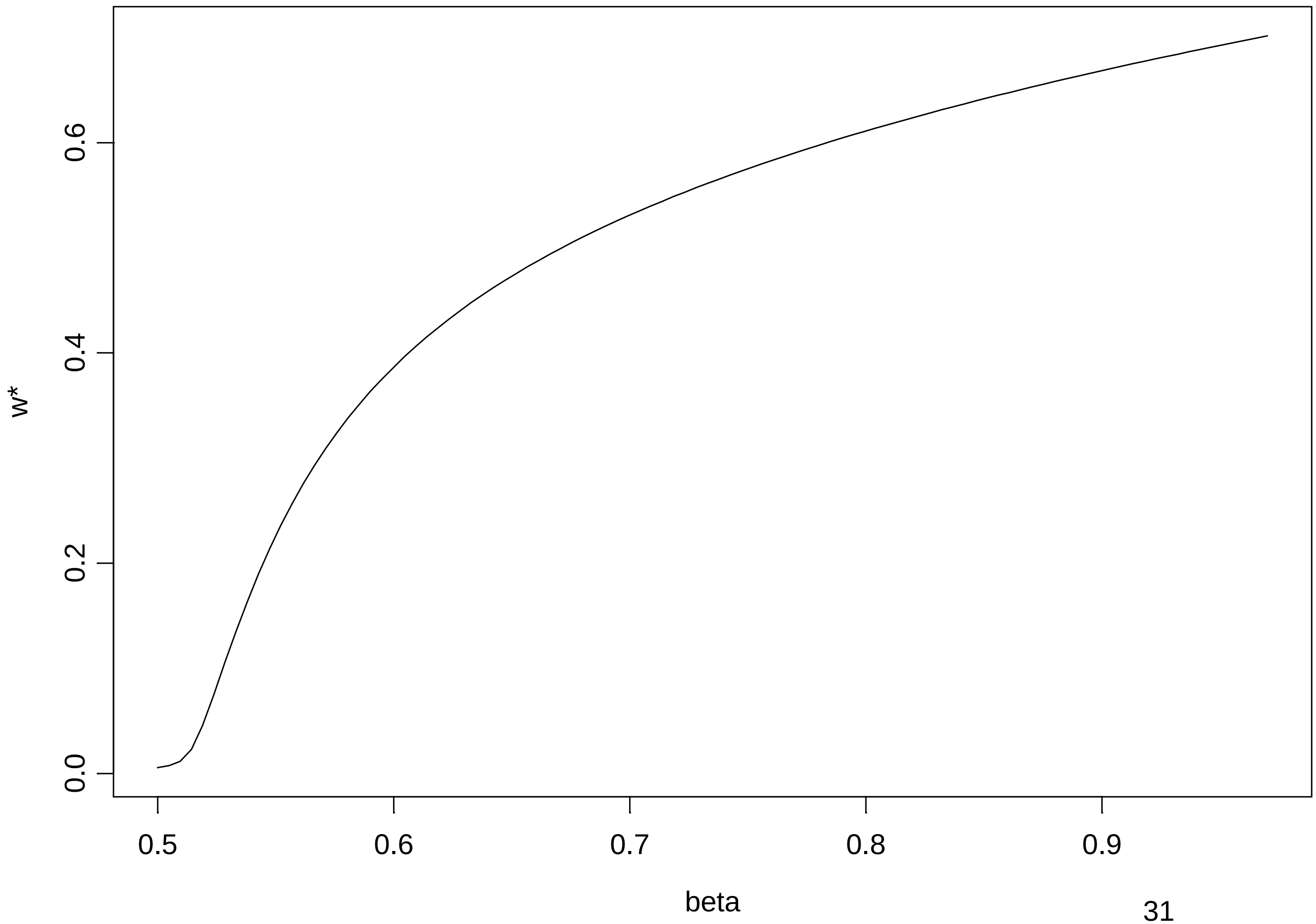
The unique positive solution

$$w^* = \sqrt{\frac{b^2 + \sqrt{b^4 + cf_0}}{2c}}$$

Recall the general solution

$$\boldsymbol{x}^* = \boldsymbol{z}_1 + w^* \boldsymbol{z}_2$$

Solutions w* of power equation



Classical mean-variance case

$$s(x) = x$$

$$\boldsymbol{f}(\mathbf{x}) = -\boldsymbol{\mu}^T \mathbf{x} + \lambda \mathbf{x}^T \boldsymbol{\Sigma} \mathbf{x} \to \inf$$

$$s_1(x) = s'(x) = 1$$

Recall general equation

$$ws_1(f_0 + b^2 w^2) = \frac{1}{2\lambda}$$

Reduces

$$w^* = \frac{1}{2\lambda}$$

The optimal portfolio

$$\boldsymbol{x}^{*} = \mathbf{z}_1 + \frac{1}{2\lambda}\mathbf{z}_2$$

Tail mean-variance optimal portfolio selection

$$s(x) = \alpha_1\sqrt{x} + \alpha_2 x$$

$$s_1(x) = s'(x) = \frac{\alpha_1}{2\sqrt{x}} + \alpha_2$$

The optimization problem reduces

$$\boldsymbol{f}(\mathbf{x}) = \boldsymbol{\mu}^T\mathbf{x} + \lambda(\alpha_1\sqrt{\mathbf{x}^T\Sigma\mathbf{x}} + \alpha_2\mathbf{x}^T\Sigma\mathbf{x}) \to \inf$$

Special case-tail mean-variance premium

$$TMV_q(-R) = TCE_q(-R) + \lambda TV_q(R)$$
$$= -\boldsymbol{\mu}^T\mathbf{x} + \lambda_{1,q}\sqrt{\mathbf{x}^T\boldsymbol{\Sigma}\mathbf{x}} + \lambda\lambda_{2,q}\mathbf{x}^T\boldsymbol{\Sigma}\mathbf{x} \rightarrow \inf$$

where

$$TV_q(R) = V(R \mid R > VaR_q(R))$$

is *tail variance* of R

General equation

$$ws_1(f_0 + b^2 w^2) = \frac{1}{2\lambda}$$

reduces to a quartic equation

$$w^4 - 2kw^3 + (f_0 + k^2 - \frac{1}{4\lambda^2})w^2 - 2kf_0 w + k^2 f_0 = 0$$

w^* -unique **real** solution of **quartic** equation on the interval [0,k]

Analysis of quartic equation

$$P_4(w) =$$

$$w^4 - 2kw^3 + (f_0 + k^2 - \frac{1}{4\lambda^2})w^2 - 2kf_0w + k^2 f_0 = 0$$

$$P_4(0) = k^2 f_0 > 0 \qquad P_4(k) = -\frac{k^2}{4\lambda^2} < 0$$

The real root exists in [0,k],
it is unique, and we found its **explicit closed form** (L. (2010))

Application to stock data returns

Consider a portfolio of 10 stocks from NASDAQ/Computers

Table 1: Portfolio mean return

Stock	ADOBE	Compuware	NVIDIA	Staples	VeriSign
Mean	0.0061	-0.0081	-0.0096	0.0058	0.0064

Stock	Sandisk	Microsoft	Citrix	Intuit	Symantec
Mean	-0.0198	0.0002	-0.0038	-0.0041	0.0061

Table 2: Portfolio covariance return

	ADOBE	Compuware	NVIDIA	Staples	VeriSign
ADOBE	0.006102	0.001173	0.000118	0.000513	0.000121
Compuware	0.001173	0.003310	0.001047	0.000498	0.000847
NVIDIA	0.000118	0.001047	0.002145	0.000122	0.000772
Staples	0.000513	0.000498	0.000122	0.002940	-0.000547
VeriSign	0.000121	0.000847	0.000772	-0.000547	0.003486

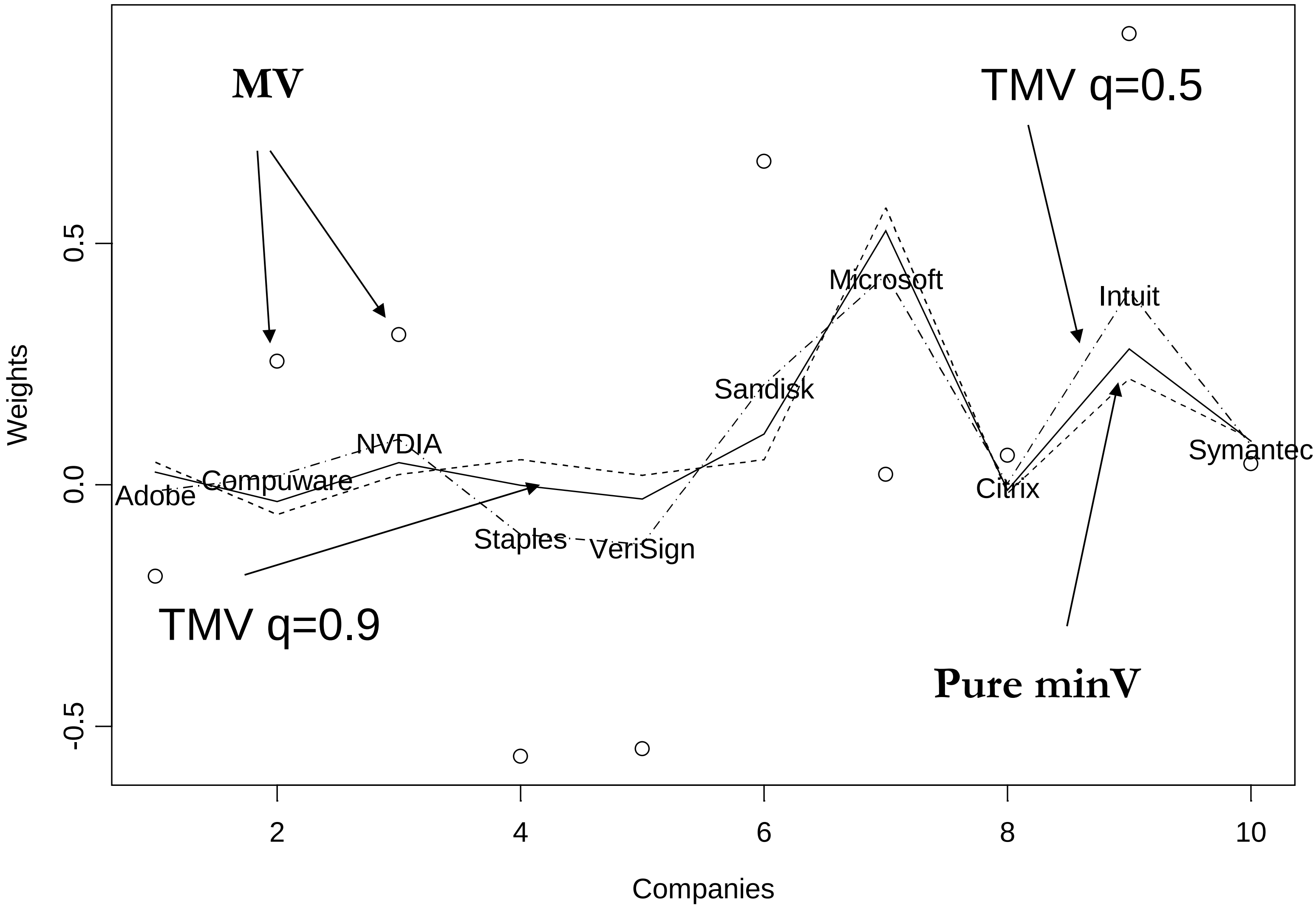
	Sandisk	Microsoft	Citrix	Intuit	Symantec
Sandisk	0.004013	-0.000033	0.000844	0.000131	0.000083
Microsoft	-0.000033	0.000485	0.000220	0.000167	0.000062
Citrix	0.000844	0.000220	0.001365	0.000397	0.000445
Intuit	0.000131	0.000167	0.000397	0.000876	0.000027
Symantec	0.000083	0.000062	0.000445	0.000027	0.002542

Optimal selection

Stock	ADOBE	Compuware	NVIDIA	Staples	VeriSign
TMV$q = 0.95$	0.026	-0.034	0.046	-0.0007	-0.029
TMV$q = 0.6$	-0.0003	0.0012	0.079	-0.07	-0.092
MV	-0.19	0.25	0.31	-0.56	-0.54

Stock	Sandisk	Microsoft	Citrix	Intuit	Symantec
TMV$q = 0.95$	0.105	0.526	-0.01	0.281	0.09
TMV$q = 0.6$	0.174	0.464	-0.002	0.36	0.08
MV	0.66	0.02	0.06	0.93	0.04

Changing portfolio with q increased



Conclusion

-Optimal selection with generalization of Sharpe ratio allows to solve many problems

This problem reduces to CFP, the solution has explicit close from and can be used for analyzing the influence of parameters of underlying distribution on the portfolio selecting

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Thank You !