Aggregate Markov models in life insurance: properties and valuation

Jamaal Ahmad[1,2]

(Joint work with Mogens Bladt[2] and Christian Furrer[2])

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$$\boldsymbol{M}(s) = \begin{pmatrix} \boldsymbol{M}_{11}(s) & \boldsymbol{M}_{12}(s) & \cdots & \boldsymbol{M}_{1J}(s) \\ \boldsymbol{M}_{21}(s) & \boldsymbol{M}_{22}(s) & \cdots & \boldsymbol{M}_{2J}(s) \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \boldsymbol{M}_{J1}(s) & \boldsymbol{M}_{J2}(s) & \cdots & \boldsymbol{M}_{JJ}(s) \end{pmatrix}$$

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- **Focal point:** Distributional properties and valuation of duration-dependent payments

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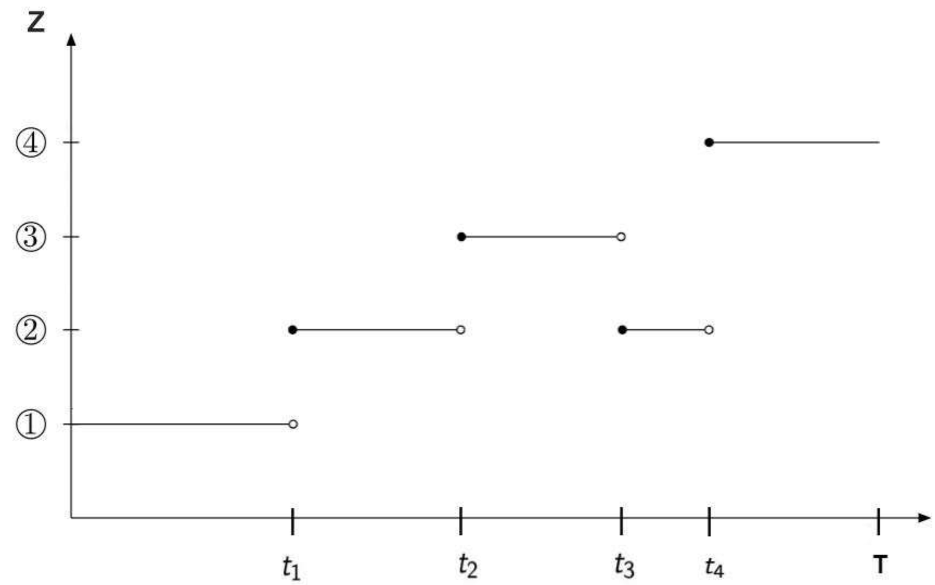

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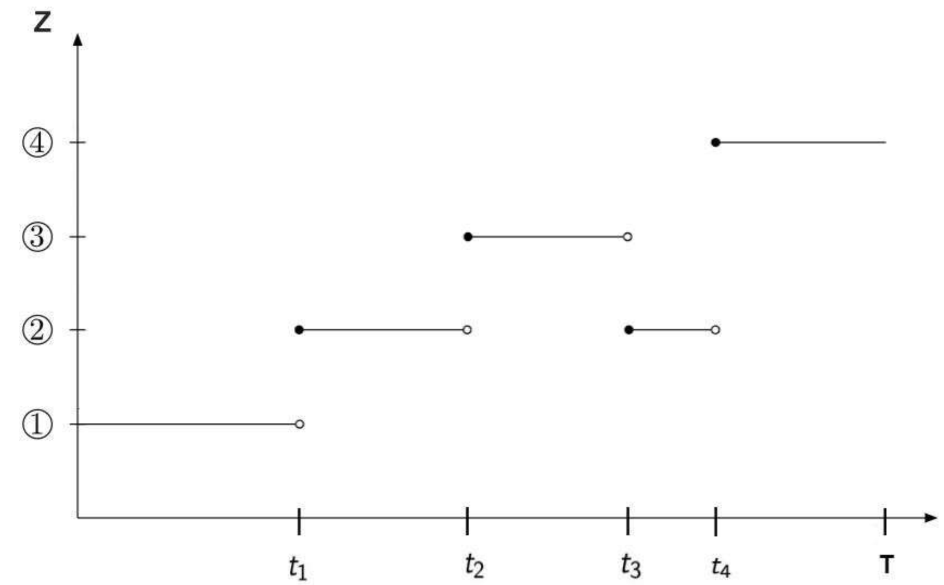
Probability of path:

$$1 \cdot e^{-\int_0^{t_1} \mu_{1\cdot}(x)\,\mathrm{d}x} \mu_{12}(t_1) \cdots e^{-\int_{t_3}^{t_4} \mu_{2\cdot}(x)\,\mathrm{d}x} \mu_{24}(t_4) e^{-\int_{t_4}^{T} \mu_{4\cdot}(x)\,\mathrm{d}x}$$



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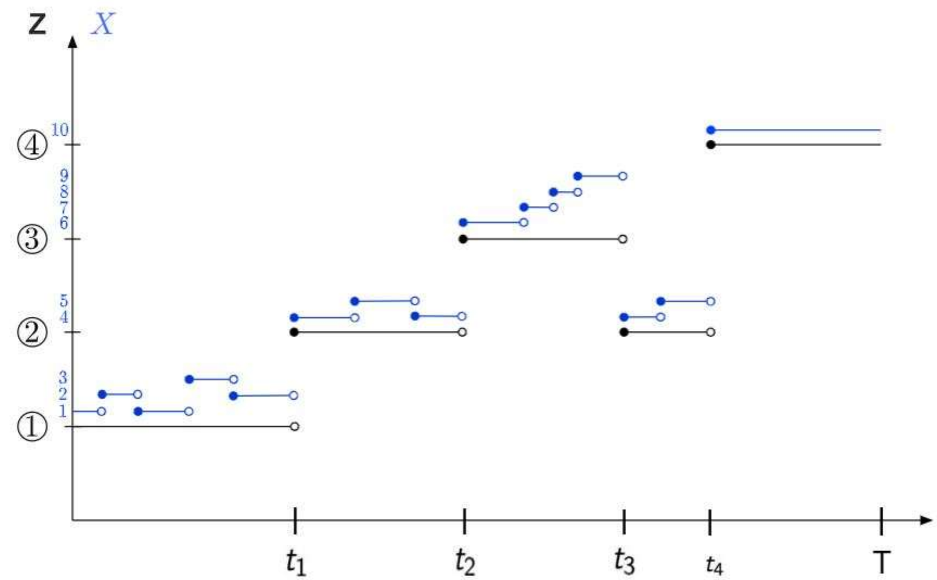

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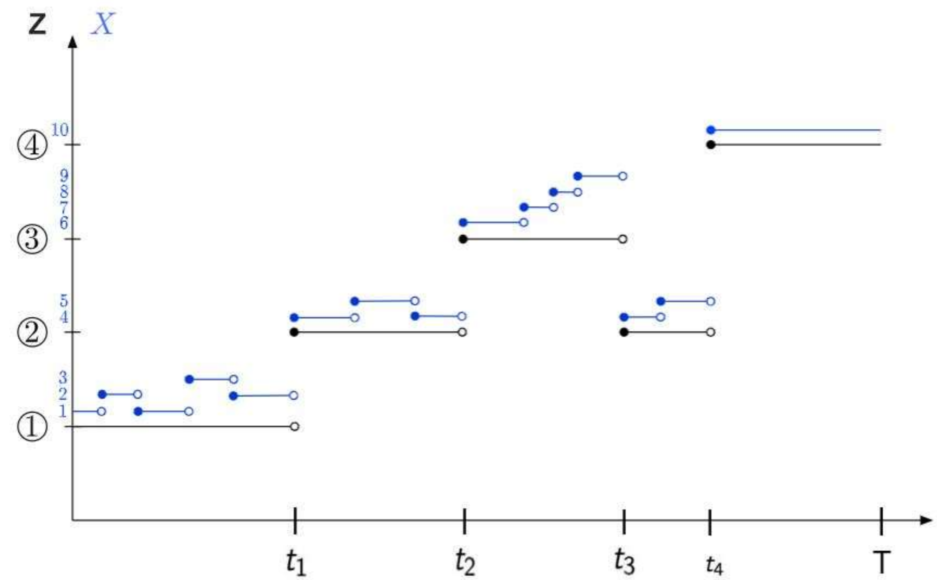
$$\boldsymbol{\pi}_1(0)\bar{\boldsymbol{P}}_{11}(0,t_1)\boldsymbol{M}_{12}(t_1)\cdots\bar{\boldsymbol{P}}_{22}(t_3,t_4)\boldsymbol{M}_{24}(t_4)\bar{\boldsymbol{P}}_{44}(t_4,T)\mathbf{1}_{d_4},$$



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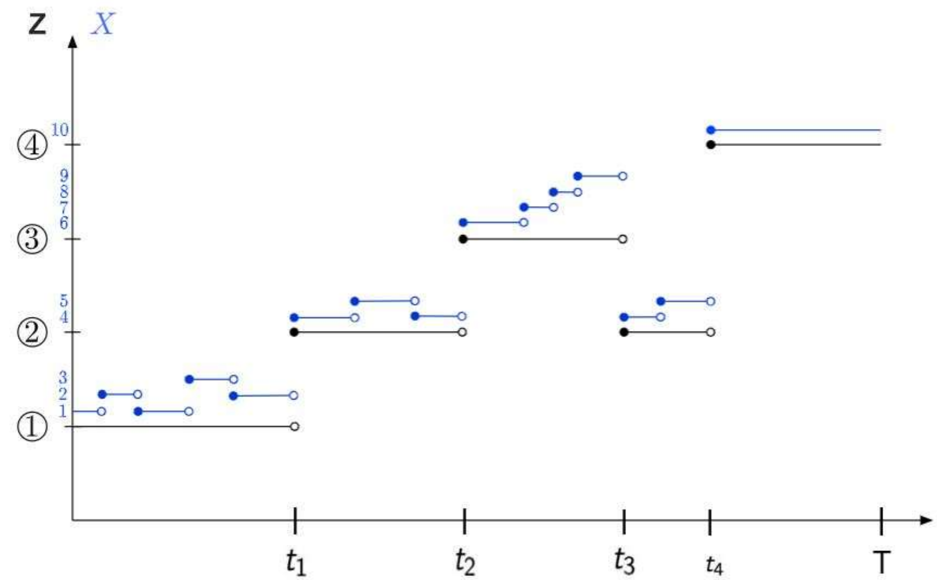

where

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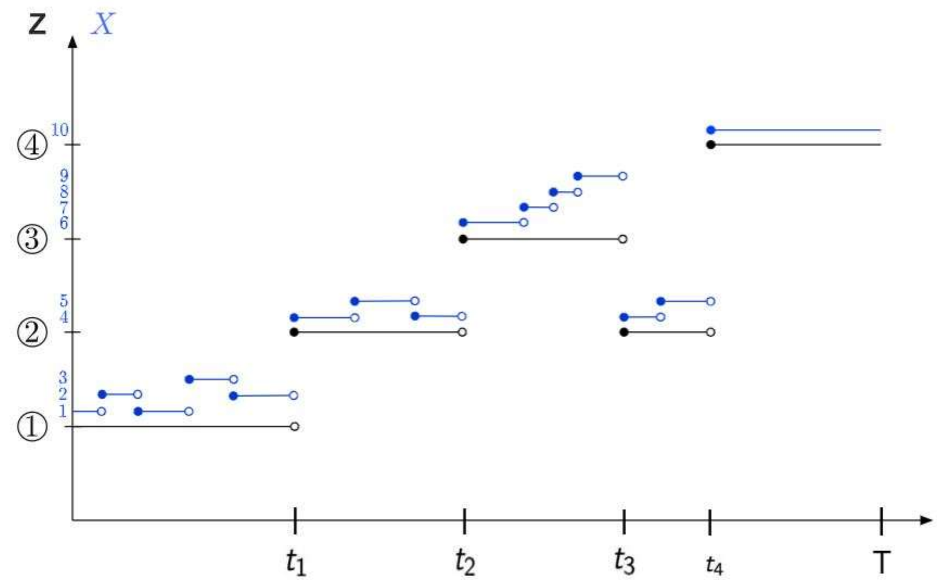

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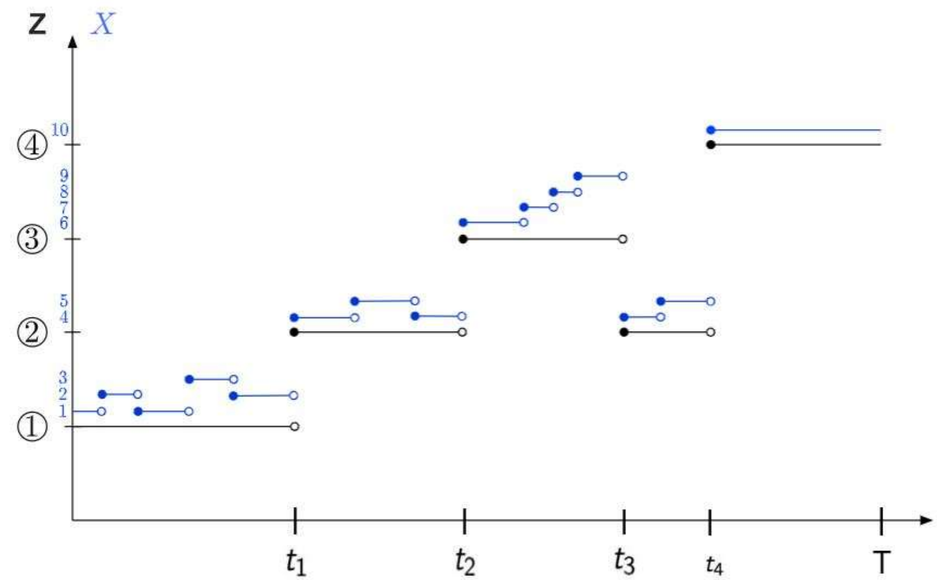

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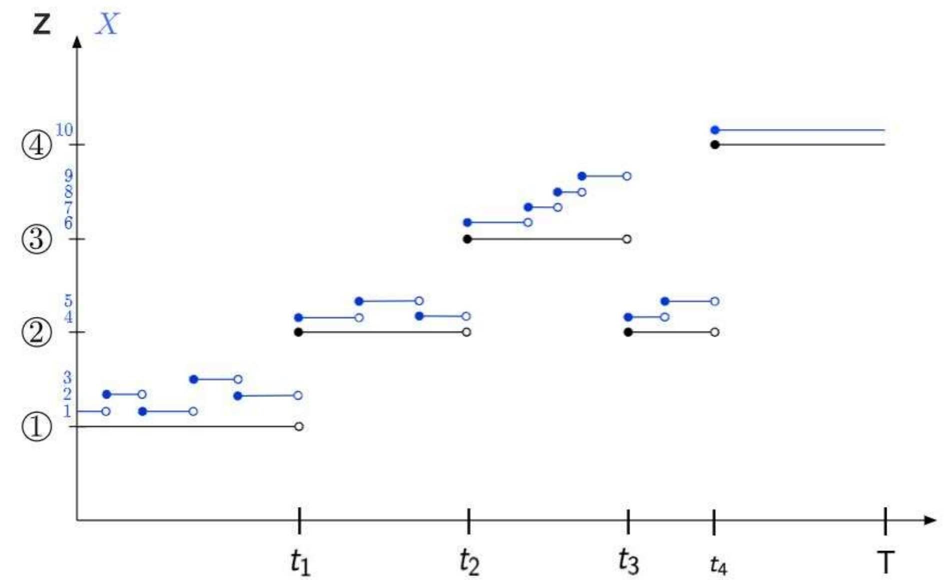

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This is a semi-Markov process with transition rates

$$\nu_{jk}(t, u) = \frac{\boldsymbol{\pi}_j(t-u)\bar{\boldsymbol{P}}_{jj}(t-u, t)}{\boldsymbol{\pi}_j(t-u)\bar{\boldsymbol{P}}_{jj}(t-u, t)\boldsymbol{e}}\boldsymbol{\beta}_{jk}(t).$$

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$$V_{i,u}(t) = \mathbb{E}\left[\int_t^T e^{-\int_t^s r(v)\mathrm{d}v} \mathrm{d}B(s) \,\middle|\, Z(t) = i, U(t) = u \right]$$

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- In our model:

$$A_{i,u}(t, \mathrm{d}s) = \sum_{\jmath \in E} \int_0^{u+s-t} p_{i\jmath}(t, u, s, \mathrm{d}z) \left(b_j(s,z) + \sum_{\substack{k \in \mathcal{J} \\ k \neq j}} b_{jk}(s,z) \beta_{\jmath k}(s) \right) \mathrm{d}s,$$

Valuation (within the reset property)

- Consider duration-dependent payment process $B = \{B(t)\}_{t \geq 0}$ (Hoem, 1972):

$$\mathrm{d}B(t) = \sum_{j \in \mathcal{J}} \left(1_{(Z(t)=j)} b_j(t, U(t))\,\mathrm{d}t + \sum_{\substack{k \in \mathcal{J} \\ k \neq j}} b_{jk}(t, U(t-))\,\mathrm{d}N_{jk}(t) \right)$$

- Prospective reserve:

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- This dimension reduction is not seen for general semi-Markov models

Numerical example

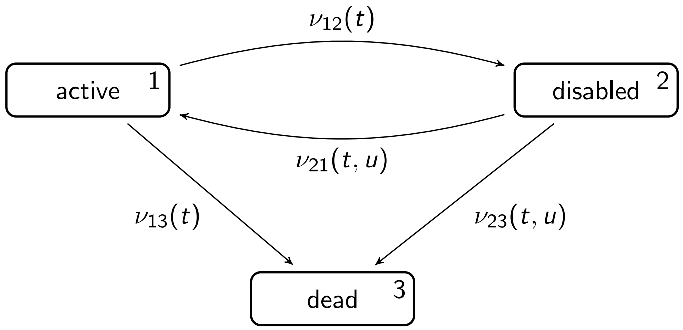
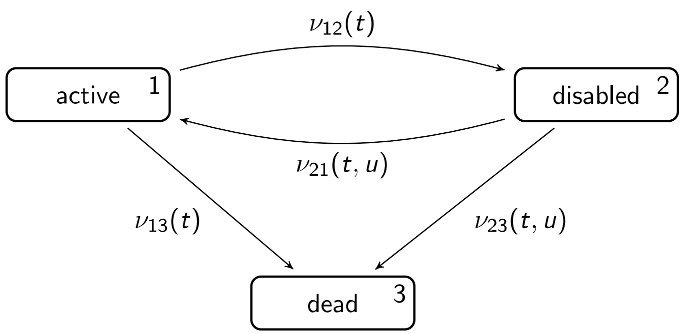


Figure: Semi-Markov disability model

Numerical example



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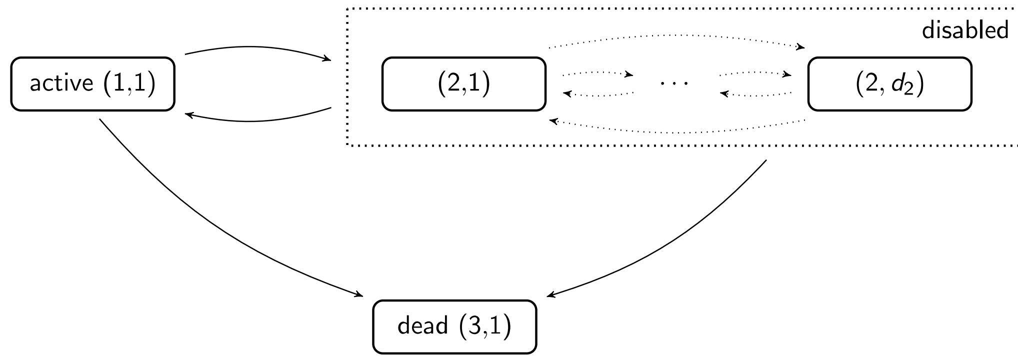
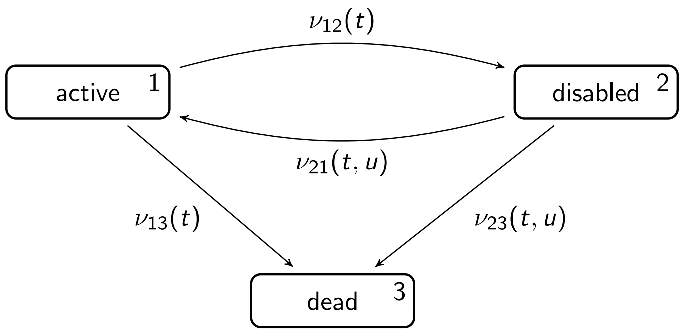


Figure: Aggregate Markov disability model

Numerical example



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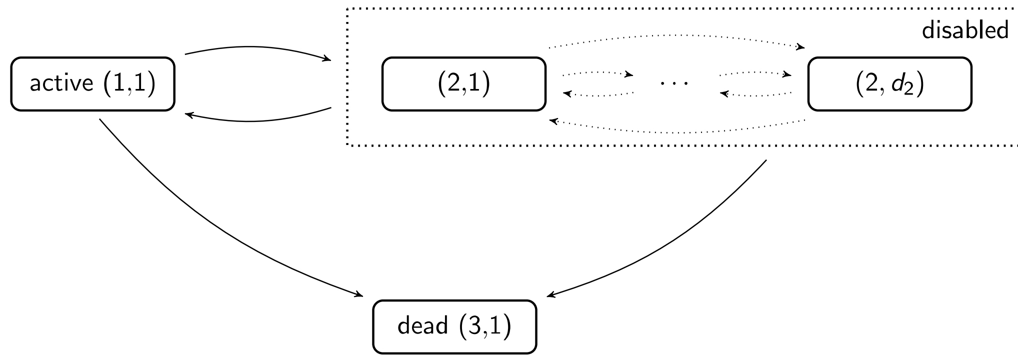
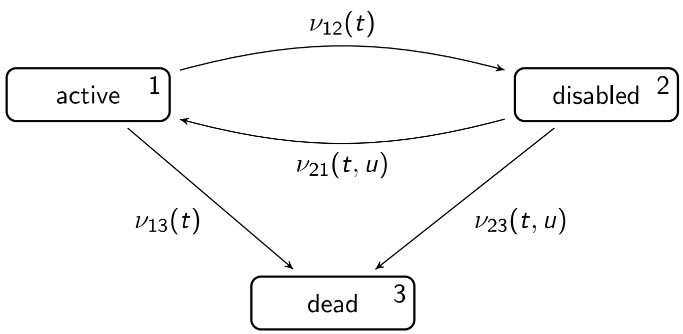


Figure: Aggregate Markov disability model

- Disability coverage with a 3 months waiting period: $b_2(s,z) = 1_{(s<65)}1_{(z>1/4)}$.

Numerical example



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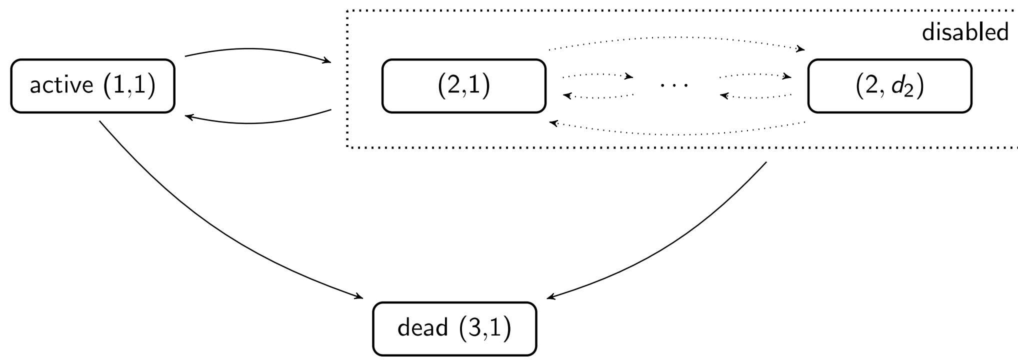
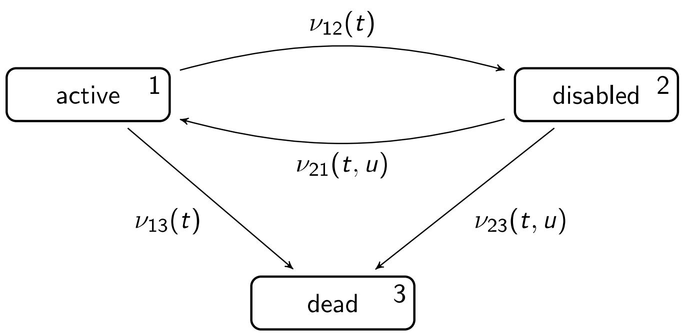


Figure: Semi-Markov disability model

Figure: Aggregate Markov disability model

- Disability coverage with a 3 months waiting period: $b_2(s, z) = 1_{(s<65)}1_{(z>1/4)}$.
- Transition rates $\nu_{ij}(t, u)$ used by a large Danish life insurance company.

Numerical example



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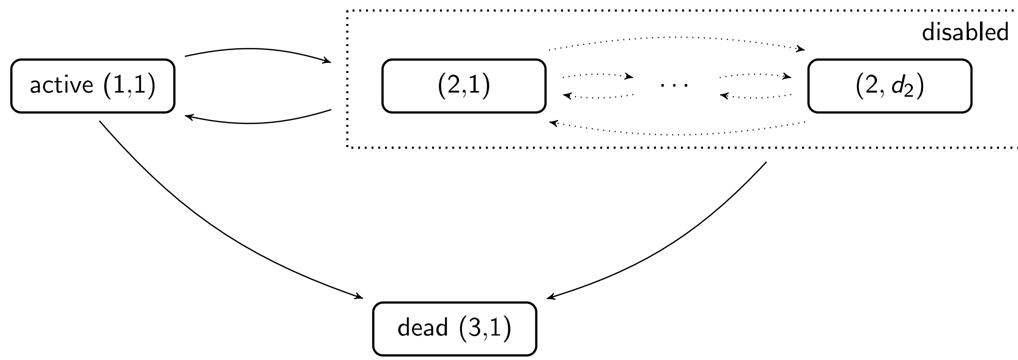

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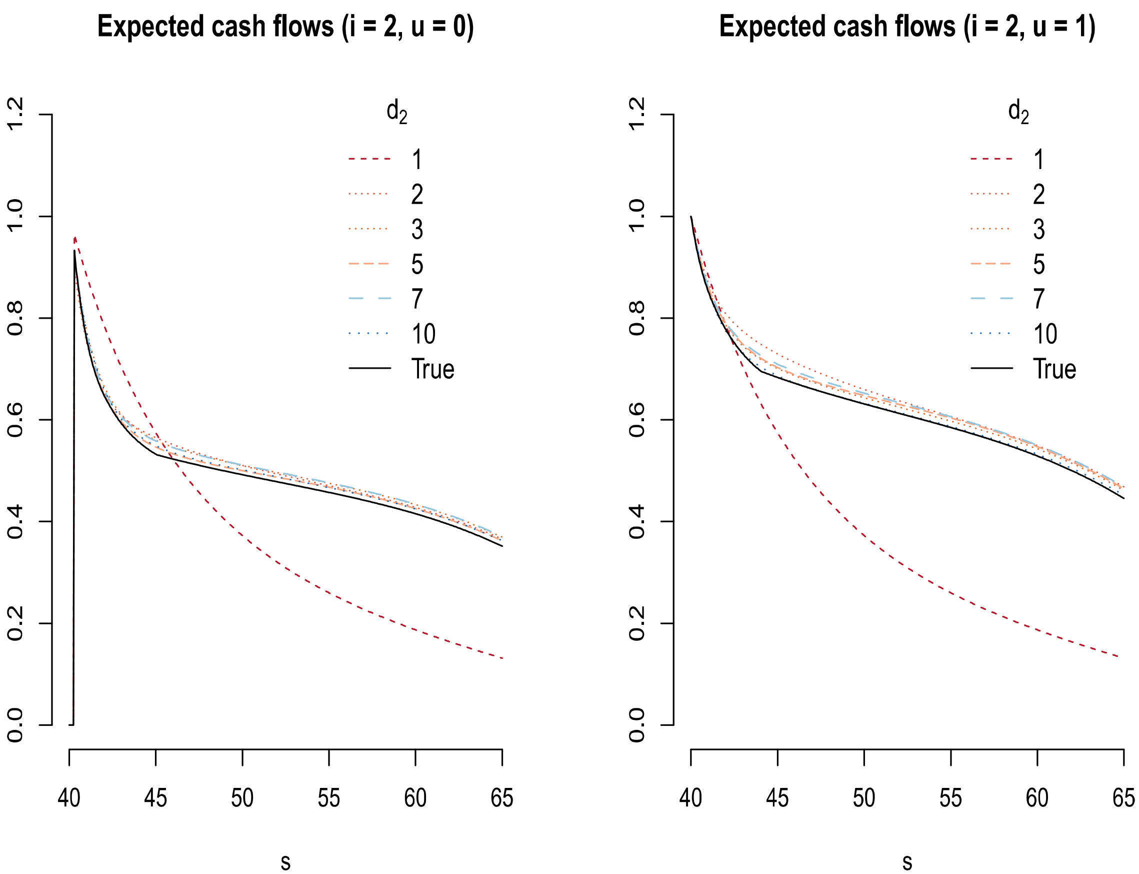
- Disability coverage with a 3 months waiting period: $b_2(s,z) = 1_{(s<65)}1_{(z>1/4)}$.
- Transition rates $\nu_{ij}(t,u)$ used by a large Danish life insurance company.
- Micro intensities fitted to simulated data using an EM algorithm developed in a companion paper Ahmad & Bladt (2022)

Cash flows and reserves



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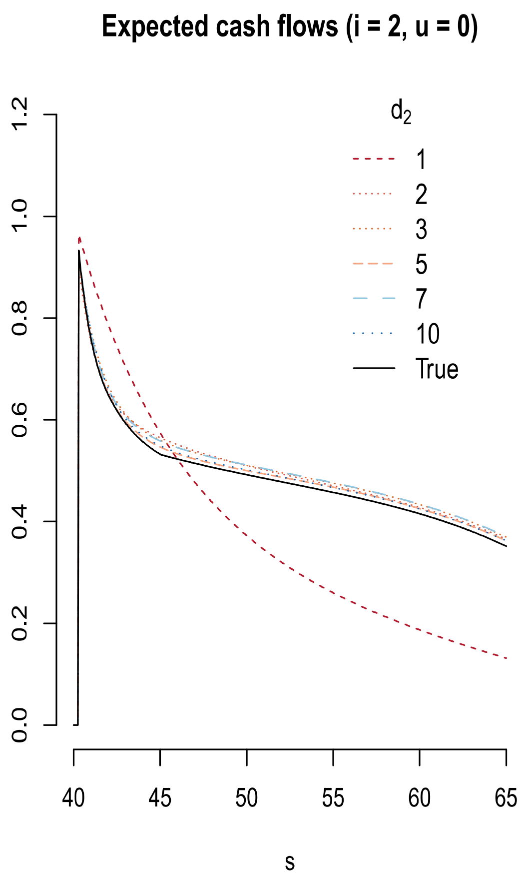

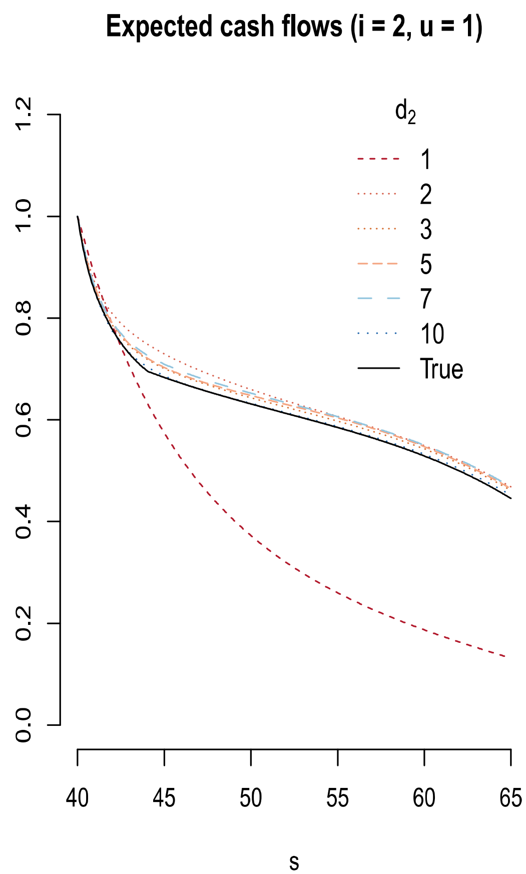

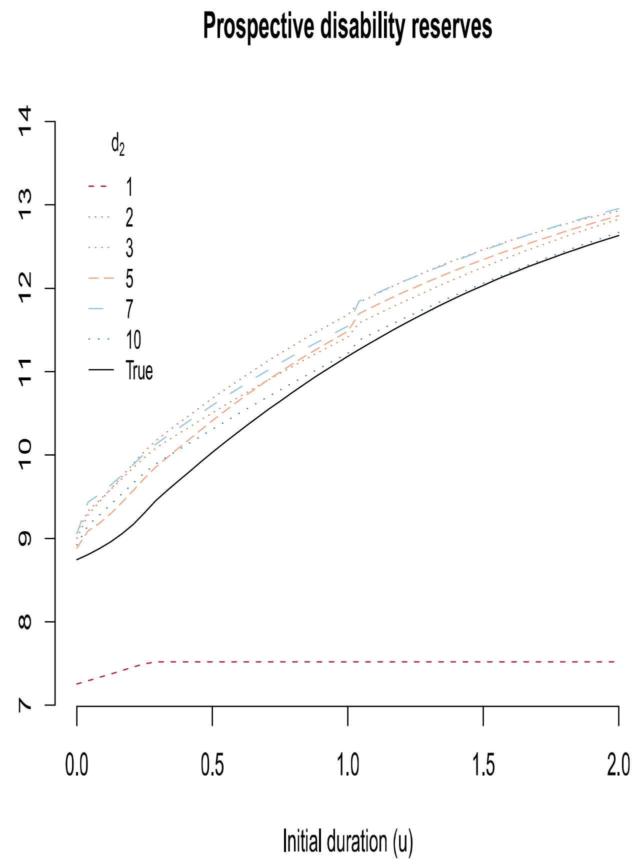

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