Pricing of Guaranteed Minimum Benefits in Variable Annuities

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Agenda

1. Introduction and motivation
2. Valuation model
3. Pricing of GMABs
4. Model calibration
5. Example
6. Conclusion & Outlook
Introduction and motivation
Variable Annuities

- **Variable Annuities** (VA) are (deferred), fund-linked annuity and insurance products allowing guaranteed payments and participation in the financial markets at the same time.

- Examples for guaranteed payments include
  - minimum interest rate guarantees
  - ratchets

- Variable annuities are often referred to as GMxB, **Guaranteed Minimum Benefits** of type x:
  - GMDB (Death)
  - GMAB (Accumulation)
  - GMIB (Income)
  - GMWB (Withdrawal)
Markets for Variable Annuities

• Motivation
  – Increasing life expectancy
  – Reduction of the state retirement pensions in several countries

• Consequences
  – VA as a major success story in the North American insurance market
  – Rapid growth of VA business in Japan - from $1.3 billion in 2001 to more than $216 billion in 2011 (assets under management)
  – Europe as the latest market for Variable Annuities

• Risks: financial, actuarial, behavioral
Existing literature

- GMDB: financial protection to dependents of the insured in case of death [Milevsky and Posner 2001], [Ulm 2008]
- GMAB: choice between fund performance and guarantee at maturity [van Haastrecht et al. 2009]
- GMIB: market value of fund account paid at once or lifelong annuity [Boyle and Hardy 2003], [Marshall et al. 2010]
- GMWB: Possibility to withdraw money from account within certain limits [Milevsky and Salisbury 2006], [Dai et al. 2008]
- General framework for pricing GMxB’s, either geometric Brownian Motion or numerical valuation: [Bauer et al. 2008], [Bacinello et al. 2011]

Our contribution:
Explicit solutions for the prices of GMABs in a hybrid model for insurance and market risk.
Valuation model
Financial market model
Notation and definitions

- \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\): filtered probability space
- \(r\): short rate process adapted to filtration \(\mathbb{F}\) and money-market account

\[
B(t) = \exp \left( \int_0^t r(s) ds \right).
\]

- \(\mathbb{Q}\): risk-neutral measure
- \(S\): traded security with \(S/B\) a \(\mathbb{Q}\)-martingale:

\[
S(t) = \mathbb{E}_\mathbb{Q} \left[ e^{-\int_t^T r(s) ds} S(T) | \mathcal{F}_t \right].
\]

- Hull-White-Black-Scholes hybrid model with time-dependent volatility (HWBS). Dynamics under \(\mathbb{Q}\):

\[
dr(t) = (\theta_r(t) - a_r(t)) dt + \sigma_r dW_r^\mathbb{Q}(t),
\]

\[
 dY(t) = \left( r(t) - \frac{1}{2} \sigma_Y^2(t) \right) dt + \sigma_Y(t) dW_Y^\mathbb{Q}(t),
\]

where \(Y(t) = \ln \left( S(t)/S(0) \right)\) and \(dW_r^\mathbb{Q}(t)dW_Y^\mathbb{Q}(t) = \rho dt\).
Insurance model
Notation and definitions

• **Random lifetime** of a person aged \( x \) at \( t = 0 \): Stopping time \( \tau_x \) of counting process \( N_{x+t}(t) \) with mortality intensity \( \lambda_{x+t}(t) \) adapted to filtration \( \mathbb{F} \).

• Mortality intensity independent from short rate and equity price.

• Introduce filtrations \( \mathbb{H} = (\mathcal{H}_t)_{t \geq 0} \) with \( \mathcal{H}_t = \sigma(\mathbbm{1}_{\{\tau_x \leq s\}} : s \leq t) \) and \( \mathcal{G} = \mathbb{F} \vee \mathbb{H} \).

• **Survival probability:**
  Probability that a person of age \( x + t \) at time \( t \) survives at least up to time \( T \):
  \[
p_{x+t}(t, T) := \mathbb{Q}(\tau_x > T | \mathcal{G}_t).
  \]

• For a person of age of \( x + t \) at time \( t \) it holds:
  \[
p_{x+t}(t, T) = \mathbb{E}_\mathbb{Q}\left[ e^{-\int_t^T \lambda_{x+s}(s) \, ds} | \mathcal{G}_t \right] = \mathbb{E}_\mathbb{Q}\left[ e^{-\int_t^T \lambda_{x+s}(s) \, ds} | \mathcal{F}_t \right].
  \]
Insurance model
Mortality improvement ratio

- Compare mortality intensity at time 0 with mortality intensity at time $t$
- Mortality improvement ratio:

$$\xi_{x+t}(t) = \frac{\lambda_{x+t}(t)}{\lambda_{x+t}(0)}$$

Sample path for the mortality improvement ratio
Insurance model
Mortality improvement ratio

- $\xi_t$ modeled as an extended Vasicek process adapted to filtration $\mathcal{F}$:
  \[ d\xi(t) = k(e^{-\gamma t} - \xi(t))dt + \sigma e_{\xi}dW^\xi(t). \]
- Initial mortality intensity described by Gompertz model:
  \[ \lambda_{x+t}(0) = \frac{1}{b} \cdot c \cdot \frac{x+t-m}{b}, \]
calibrated to the current life table.
- Future mortality intensity can be calculated by
  \[ \lambda_{x+t}(t) = \lambda_{x+t}(0) \cdot \xi(t). \]
- Survival probability can be expressed as:
  \[ p_{x+t}(t, T) = C_\lambda(t, T)e^{-D_\lambda(t,T)\lambda_{x+t}(t)}, \]
  where $C_\lambda(t, T)$ and $D_\lambda(t, T)$ satisfy two ordinary differential equations which can be solved analytically.
Pricing of variable annuities
Guaranteed Minimum Accumulation Benefit

Definition

- IP: single premium
- \( A(t) \): account value at time \( t \), \( A(0) = IP \), 100% invested in equities.
- \( G(T) \): guaranteed amount at end of the accumulation period \( T \)
- GMAB provides policyholder, who is alive at \( T \), with a benefit \( V(T) \):
  \[
  V(T) = 1_{\{\tau > T\}} \cdot \max(A(T), G(T))
  \]
- Common options for \( G(T) \):
  - Return of premium: \( G(T) = IP \)
  - Roll-up \( G(T) = IP \cdot e^{\delta T} \), with continously compounded roll-up rate \( \delta \)
  - Ratchet \( G(T) = \max_{t_i < T} A(t_i) \)
- Fair value of GMAB at \( t = 0 \):
  \[
  V(0) = \mathbb{E}_Q \left[ e^{-\int_0^T r(s) ds} 1_{\{\tau > T\}} \max(A(T), G(T)) \right]
  \]
Guaranteed Minimum Accumulation Benefit
Roll-up guarantee

Theorem 1.
Explicit expression for $V(0)$ with $G(T) = IP \cdot e^{\delta T}$:

$$V(0) = IP \cdot p_x(0, T) \cdot \Phi \left( \frac{\mu^S_Y(T) - \delta T}{\sigma^S_Y(T)} \right)$$

$$+ \ IP \cdot P^m(0, T) \cdot e^{\delta T} \cdot \Phi \left( \frac{\delta T - \mu^T_Y(T)}{\sigma^T_Y(T)} \right),$$

with

- $\Phi$: distribution function of a standard normal distribution
- **Mortality-adjusted zero-coupon bond:**
  $$P^m(0, T) = P(0, T) \cdot p_x(0, T).$$
- $\mu^S_Y(T), \sigma^S_Y(T)$ are the moments under the equity measure $Q^S$
- $\mu^T_Y(T), \sigma^T_Y(T)$ are the moments under the forward measure $Q^T$
Theorem 2.

Explicit expression for $V(0)$ with $G(T) = \max_{t_i < T} A(t_i)$:

$$V(0) = \mathbf{1} \mathbf{P} \cdot p_x(0, T) \cdot \left( \Phi_{n-1}(0; -\mu_{\Delta_k Y}^S, \Sigma_{\Delta_k Y}^S) \right. $$

$$+ \sum_{k=1}^{n-1} \left( \Phi_{n-1}(0; -\mu_{\Delta_k Y}^S - \Sigma_{\Delta_k Y}^S e_{n-1}, \Sigma_{\Delta_k Y}^S) \right) \cdot e^{\mu_{\Delta_{n,k} Y}^S + \frac{(\sigma_{\Delta_{n,k} Y}^S)^2}{2}},$$

with

• $e_k$: unit vector with $k$-th element equal to 1

• $\mu_{\Delta_k Y}^S, \Sigma_{\Delta_k Y}^S$ are the mean vector and covariance matrix under $Q^S$ of

$$\Delta_k Y := \{\Delta_{i,k} Y\}_{i \in \{1,\ldots,n\} \setminus \{k\}}$$

with

$$\Delta_{i,k} Y := \{Y(t_k) - Y(t_i)\}_{i \in \{1,\ldots,n\} \setminus \{k\}}, \quad t_n := T$$

• $\Phi_{n-1}(u, \mu, \Sigma)$: multivariate normal distribution function with mean vector $\mu$ and covariance matrix $\Sigma$. 

Guaranteed Minimum Accumulation Benefit
Ratchet guarantee
Guaranteed Minimum Accumulation Benefit
Ratchet guarantee

Proof.

- Separate insurance and financial parts and rewrite expectation:

\[
\begin{align*}
V(0) &= \mathbb{E}_Q \left[ e^{-\int_0^T r(s) \, ds} \cdot 1_{\tau > T} \cdot \max_{t_i} \left( A(T), \max_{t_i} A(t_i) \right) \right] \\
&= \mathbb{E}_Q \left[ 1_{\tau > T} \right] \cdot \mathbb{E}_Q \left[ e^{-\int_0^T r(s) \, ds} \cdot \max_{t_i} \left( A(T), \max_{t_i} A(t_i) \right) \right] \\
&= p_x(0, T') \cdot \sum_{k=1}^{n} \mathbb{E}_Q \left[ e^{-\int_0^T r(s) \, ds} \cdot A(t_k) \cdot 1_{A(t_k) \geq A(t_i), i \in \{1, \ldots, n\} \setminus \{k\}} \right] \\
&= p_x(0, T') \cdot \left( \sum_{k=1}^{n} I_{t_k} \right) \\
\end{align*}
\]

with

\[
I_{t_k} := \mathbb{E}_Q \left[ e^{-\int_0^T r(s) \, ds} \cdot A(t_k) \cdot 1_{A(t_k) \geq A(t_i), i \in \{1, \ldots, n\} \setminus \{k\}} \right] .
\]
Guaranteed Minimum Accumulation Benefit
Ratchet guarantee

Proof (continued).

- Change to equity measure:

\[ I_{tk} = \mathbb{E}_Q \left[ e^{-\int_0^T r(s)ds} \cdot A(t_n) \cdot \frac{A(t_k)}{A(t_n)} \cdot \mathbb{1}_{A(T) \geq A(t_i), i \in \{1,\ldots,n\}\{k\}} \right] \]

\[ = A(0) \cdot \mathbb{E}_Q \left[ \frac{A(t_k)}{A(t_n)} \cdot \mathbb{1}_{\frac{A(t_i)}{A(t_k)} \leq 1, i \in \{1,\ldots,n\}\{k\}} \right] \]

\[ = A(0) \cdot \mathbb{E}_Q \left[ e^{Y(t_k) - Y(t_n)} \cdot \mathbb{1}_{Y(t_i) - Y(t_k) \leq 0, i \in \{1,\ldots,n\}\{k\}} \right] \]

\[ = A(0) \cdot \mathbb{E}_Q \left[ e^{\Delta_{nk}Y} \cdot \mathbb{1}_{\Delta_{ki}Y \leq 0, i \in \{1,\ldots,n\}\{k\}} \right] \]

with

\[ \Delta_{ij}Y = Y(t_j) - Y(t_i), \quad t_n := T. \]

- Integration over multivariate normal density function gives final formula.
Model calibration
Insurance model calibration

Data

- Initial mortality table (Source: Federal Statistical Office of Germany)

![Mortality rates graph]

- Mortality improvement ratio (Source: Federal Statistical Office of Germany)

![Mortality improvement ratio graph]
Insurance model calibration
Algorithm and results

- Gompertz model: via least-squares method.
- Mortality improvement ratio: via maximum likelihood method.
- Log-likelihood function:
  \[
  \mathcal{L}(k, \gamma, \sigma_\xi) = \sum_{i=1}^{n} \ln(f(\xi_i|\xi_{i-1}; k, \gamma, \sigma_\xi))
  = \frac{n}{2} \ln(2\pi) - n \ln \hat{\sigma}_\xi 
  - \frac{1}{2\hat{\sigma}_\xi^2} \sum_{i=1}^{n} \left( \xi_i - \xi_{i-1}e^{-k\cdot\Delta} - \frac{k}{k - \gamma}e^{-\gamma t_i} \cdot \left(1 - e^{-(\gamma-k)\cdot\Delta}\right) \right)^2,
  \]

  where
  \[
  \hat{\sigma}_\xi = \sigma_\xi \sqrt{\frac{1 - e^{-2k\cdot\Delta}}{2k}}
  \]
- Result:

<table>
<thead>
<tr>
<th>Mortality</th>
<th>b</th>
<th>m</th>
<th>k</th>
<th>\gamma</th>
<th>\sigma_\xi</th>
</tr>
</thead>
<tbody>
<tr>
<td>female</td>
<td>7.80</td>
<td>88.09</td>
<td>0.5529</td>
<td>0.0223</td>
<td>0.0512</td>
</tr>
<tr>
<td>male</td>
<td>9.57</td>
<td>83.89</td>
<td>0.4301</td>
<td>0.0179</td>
<td>0.0485</td>
</tr>
</tbody>
</table>
Financial model calibration

Data

- Interest rate data: deposit rates, swaps, swaptions (Source: Bloomberg)

- Equity data: implied volatilities term structure (Source: Bloomberg)
Financial model calibration
Algorithm

- \( \theta_r(t) \): shift to current term structure of interest rates

- Hull-White model: minimize sum of squared deviations from observed European swaption prices

- Result: \( a_r = 0.0151 \) and \( \sigma_r = 0.009 \).

- Instantaneous volatility: (piecewise) constant, extracted by recursion.

- Correlation: historical correlation between EuroStoxx50 log-returns and absolute differences in 3-month zero rates.

- Result: \( \sigma_S = 0.2923 \) and \( \rho = 0.1209 \).
Example 5
Setup

- Type of the guarantee: single premium GMAB, $T = 20$ years.
- Maturity of the guarantee: 20 years.
- Policyholder: male, 45 years old.
- Mortality improvement ratio: German population for period 1968-2008.
- Roll-up and ratchet considered:

Ratchet step = 4 years

Roll-up rate = 2%
Sensitivities to product parameters

Roll-up rate

- Roll-up guarantee

<table>
<thead>
<tr>
<th>Roll-up</th>
<th>Roll-up rate</th>
<th>GMAB</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0%</td>
<td>102.49</td>
</tr>
<tr>
<td>2</td>
<td>1.5%</td>
<td>111.51</td>
</tr>
<tr>
<td>3</td>
<td>3.0%</td>
<td>125.64</td>
</tr>
</tbody>
</table>

- Ratchet guarantee

<table>
<thead>
<tr>
<th>Ratchet</th>
<th>Ratchet step</th>
<th>GMAB</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2 years</td>
<td>125.28</td>
</tr>
<tr>
<td>2</td>
<td>4 years</td>
<td>118.49</td>
</tr>
<tr>
<td>3</td>
<td>8 years</td>
<td>114.19</td>
</tr>
</tbody>
</table>
Sensitivities to financial market parameters
Equity volatility

- Sensitivities: Central difference quotient for a parallel shift of ±0.01%.
- Stress test according to QIS5 calibration paper for Solvency II\(^a\):
  Relative increase (up stress) of 50% and decrease (down stress) of 15% from current value.
- Roll-up guarantee

<table>
<thead>
<tr>
<th>ImpVol</th>
<th>Roll-Up 1</th>
<th>Roll-Up 2</th>
<th>Roll-Up 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sensitivity</td>
<td>0.75%</td>
<td>0.98%</td>
<td>1.19%</td>
</tr>
<tr>
<td>Current value</td>
<td>102.49</td>
<td>111.51</td>
<td>125.64</td>
</tr>
<tr>
<td>Up stress</td>
<td>111.66</td>
<td>122.99</td>
<td>139.43</td>
</tr>
<tr>
<td>Down stress</td>
<td>99.69</td>
<td>107.83</td>
<td>121.13</td>
</tr>
</tbody>
</table>

- Ratchet guarantee

<table>
<thead>
<tr>
<th>ImpVol</th>
<th>Ratchet 1</th>
<th>Ratchet 2</th>
<th>Ratchet 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sensitivity</td>
<td>2.24%</td>
<td>1.81%</td>
<td>1.24%</td>
</tr>
<tr>
<td>Current value</td>
<td>125.28</td>
<td>118.49</td>
<td>114.19</td>
</tr>
<tr>
<td>Up stress</td>
<td>155.89</td>
<td>142.53</td>
<td>133.53</td>
</tr>
<tr>
<td>Down stress</td>
<td>117.14</td>
<td>111.93</td>
<td>108.74</td>
</tr>
</tbody>
</table>

\(^a\) Committee of the European Insurance and Occupational Pension Supervisors, CEIOPS-SEC-40-10.
Sensitivities to financial market parameters

**Interest rates**

- Sensitivities: Central difference quotient for a parallel shift of \( \pm 0.01\% \).
- Stress test scenarios according to QIS5 calibration paper for Solvency II\(^a\):
- Roll-up guarantee

<table>
<thead>
<tr>
<th>IR</th>
<th>Roll-up 1</th>
<th>Roll-up 2</th>
<th>Roll-up 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sensitivity</td>
<td>-4.19%</td>
<td>-6.73%</td>
<td>-10.44%</td>
</tr>
<tr>
<td>Current value</td>
<td>102.49</td>
<td>111.51</td>
<td>125.64</td>
</tr>
<tr>
<td>Up stress</td>
<td>98.90</td>
<td>105.73</td>
<td>116.63</td>
</tr>
<tr>
<td>Down stress</td>
<td>107.96</td>
<td>120.13</td>
<td>138.80</td>
</tr>
</tbody>
</table>

- Ratchet guarantee

<table>
<thead>
<tr>
<th>IR</th>
<th>Ratchet 1</th>
<th>Ratchet 2</th>
<th>Ratchet 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sensitivity</td>
<td>-6.76%</td>
<td>-6.16%</td>
<td>-4.74%</td>
</tr>
<tr>
<td>Current value</td>
<td>125.28</td>
<td>118.49</td>
<td>114.19</td>
</tr>
<tr>
<td>Up stress</td>
<td>120.95</td>
<td>114.35</td>
<td>110.49</td>
</tr>
<tr>
<td>Down stress</td>
<td>133.92</td>
<td>126.33</td>
<td>121.12</td>
</tr>
</tbody>
</table>

\(^a\) The altered term structures are derived by multiplying the current interest rate curve by \( 1 + s_{\text{up}} \) and \( 1 + s_{\text{down}} \), where \( s_{\text{up}} \) (\( s_{\text{down}} \)) ranges from 0.70 (\( -0.75 \)) for short-term maturities to 0.25 (\( -0.30 \)) for long-term maturities.
Sensitivities to insurance market parameters

Mortality

- Sensitivities: one-directional difference quotient for a relative decrease of 1%.

- Stress test according to Solvency II requirements: 25% reduction applied to entire mortality table.

- Roll-up guarantee

<table>
<thead>
<tr>
<th>Mortality</th>
<th>Roll-up 1</th>
<th>Roll-up 2</th>
<th>Roll-up 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sensitivity</td>
<td>0.11%</td>
<td>0.12%</td>
<td>0.14%</td>
</tr>
<tr>
<td>Initial</td>
<td>102.49</td>
<td>111.51</td>
<td>125.64</td>
</tr>
<tr>
<td>Reduced</td>
<td>105.32</td>
<td>114.56</td>
<td>129.10</td>
</tr>
</tbody>
</table>

- Ratchet guarantee

<table>
<thead>
<tr>
<th>Mortality</th>
<th>Ratchet 1</th>
<th>Ratchet 2</th>
<th>Ratchet 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sensitivity</td>
<td>0.14%</td>
<td>0.13%</td>
<td>0.08%</td>
</tr>
<tr>
<td>Initial</td>
<td>125.28</td>
<td>118.49</td>
<td>114.19</td>
</tr>
<tr>
<td>Reduced</td>
<td>128.72</td>
<td>121.76</td>
<td>117.33</td>
</tr>
</tbody>
</table>
Conclusion & Outlook
Conclusion & further research

- HWBS for the financial market.
- 2-step approach for stochastic mortality modelling.
- Explicit expressions for GMABs with different guarantee riders.
- Calibration of the presented hybrid model.
- Example with sensitivity analysis.

- Analyse other types of guarantees (GMIB, GMDB).
- Incorporate policyholder behavior risk.
  (with Escobar, M., Ramsauer, F., Saunders, D., Zagst, R.)
Thank you for your attention.
Bibliography


Appendix
Zero-coupon bond

- Zero-coupon bond:

\[ P(t, T) = \mathbb{E}_Q \left[ e^{-\int_t^T r(u) du} | \mathcal{F}_t \right] = C_r(t, T) \cdot e^{-D_r(t, T)r(t)} \]

with

\[ C_r(t, T) = \frac{P^M(0, T)}{P^M(0, t)} \cdot \exp \left[ D_r(t, T) f^M(0, t) - \frac{\sigma_r^2}{4a} (1 - e^{-2a_r t}) D_r(t, T)^2 \right] \]

\[ D_r(t, T) = \frac{1}{a_r} \left[ 1 - e^{a_r(t-T)} \right] \]

- Long-term zero-coupon rate \( R(t, T) \) is a linear function of short rate \( r(t) \):

\[ R(t, T) = -a + br(t), \]

with

\[ a := \log(C_r(t, T))/(T - t) \text{ and } b := D_r(t, T)/(T - t). \]
Appendix

Zero-coupon bond as a numeraire

- \( Q^T \): **T-forward measure** with zero-coupon bond \( P(\cdot, T) \) as numeraire.

- Corresponding Radon-Nikodym derivative:
  \[
  \frac{dQ^T}{dQ} = \frac{P(T, T)/P(t, T)}{B(T)/B(t)} = \exp \left[ -\frac{1}{2} \int_0^T \gamma^2(t) dt - \int_0^T \gamma(t) dW^Q_r \right],
  \]
  with
  \[ \gamma(t) = \sigma_r \cdot D_r(t, T). \]

- Dynamics under \( Q^T \):
  \[
  dr(t) = (\theta_r(t) - a_r r(t) - \sigma_r^2 D_r(t, T)) dt + \sigma_r dW^Q_r(t),
  \]
  \[
  dY(t) = \left( r(t) - \frac{1}{2} \sigma_Y^2(t) - \sigma_Y(t) \sigma_r \rho D_r(t, T) \right) dt + \sigma_Y(t) dW^Q_Y(t).
  \]

- \( r(T) \) and \( Y(T) \) are normally distributed with corresponding moments
  \[ \mu_{r(T)}^{Q^T}, \sigma_{r(T)}^{Q^T} \] and \[ \mu_{Y(T)}^{Q^T}, \sigma_{Y(T)}^{Q^T} \].
Appendix
Equity price as a numeraire

- **$\mathcal{Q}^S$: equity measure** with equity price $S$ as numeraire.
- Corresponding Radon-Nikodym derivative:
  \[
  \frac{d\mathcal{Q}^S}{d\mathcal{Q}} = \frac{S(T)/S(t)}{B(T)/B(t)} = \exp \left[ -\frac{1}{2} \int_0^T \sigma_Y^2(t) dt + \int_0^T \sigma_Y(t) dW_Y(t) \right],
  \]
- Dynamics under $\mathcal{Q}^S$:
  \[
  dr(t) = \left( \theta_r(t) - a_r r(t) + \sigma_r \sigma_Y(t) \rho \right) dt + \sigma_r dW_r^{\mathcal{Q}^S}(t),
  \]
  \[
  dY(t) = \left( r(t) + \frac{1}{2} \sigma_Y^2(t) \right) dt + \sigma_Y(t) dW_Y^{\mathcal{Q}^S}(t).
  \]
- $r(T)$ and $Y(T)$ are normally distributed with corresponding moments
  \[
  \mu_{r(T)}^{\mathcal{Q}^S}, \sigma_{r(T)}^{\mathcal{Q}^S} \quad \text{and} \quad \mu_{Y(T)}^{\mathcal{Q}^S}, \sigma_{Y(T)}^{\mathcal{Q}^S}.
  \]